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The spectrum of certain singular integral operators on symmetric spaces

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ABSTRACT

Relevance of research. The study of the spectral properties of integral and singular integral operators defined on various function spaces is one of the principal areas of modern functional analysis and operator theory. The spectrum of an operator characterizes the existence of its inverse, the analytic structure of its resolvent, the solvability of operator equations, and the applicability of functional calculus. Thus, determining the spectrum provides important information not only about an individual operator but also about the geometric and interpolation properties of the space on which it acts.

Among singular integral operators, the Hilbert transform is a classical object of harmonic analysis, Fourier analysis, complex analysis, and the theory of singular integral equations. Although its boundedness and spectrum in Lebesgue spaces have been studied extensively, the spectral nature of the operator becomes more complicated when one passes from Lebesgue spaces to broader classes of function spaces. In particular, while the Hilbert transforms on the real line, the half-line, and a finite interval are defined by kernels of the same type, their spectra have different geometric structures. This distinction is related to the geometry of the domain, the properties of the rearrangement-invariant space, and its Boyd indices.

Rearrangement-invariant spaces provide a unified theoretical framework for Lebesgue, Lorentz, Orlicz, and many other classical spaces. In such spaces, studying not only the boundedness of operators but also their spectra makes it possible to extend known results from individual spaces to entire classes of function spaces. Moreover, the lower and upper Boyd indices describe the asymptotic behavior of dilation operators and play a central role in estimating the resolvents of singular integral operators. For this reason, investigating the spectrum of the Hilbert transform in rearrangement-invariant and Lorentz spaces is an important problem in contemporary operator theory.

The second class of operators considered in the dissertation consists of generalized Cesàro operators, which extend the classical Cesàro operator. These operators arise in the theory of integral operators, Hardy-type inequalities, averaging processes, and operator semigroup theory. Although their properties in Lebesgue spaces and certain Sobolev spaces are known, their spectral characteristics in

Lorentz spaces and general rearrangement-invariant spaces have not been fully investigated. Representing generalized Cesàro operators by means of special strongly continuous groups and semigroups, identifying their generators, and describing their spectra through functional calculus provide a new and effective approach to this problem.

The relevance of the dissertation lies in the study of two important classes of operators—the Hilbert transform and generalized Cesàro operators—in rearrangement-invariant and Lorentz spaces. The work reveals how the spectrum depends on the domain of definition, the structure of the underlying space, and its Boyd indices, and it uses the connections among Banach algebras, interpolation theory, strongly continuous groups and semigroups, and functional calculus.

Aim of the research. To investigate the spectral and resolvent properties of the Hilbert transform and generalized Cesàro operators defined on rearrangement-invariant Banach function spaces and Lorentz spaces, and to characterize their spectra in terms of the domain of definition and the structural parameters of the underlying space.

Research objectives. To achieve this aim, the following objectives were set: to systematize the necessary concepts concerning rearrangement-invariant spaces, Lorentz spaces, Boyd indices, and operator spectra; to determine the spectrum and resolvent of the Hilbert transform on the real line and the half-line in Lorentz spaces; to investigate the boundedness of the weighted Hilbert transform on the half-line in rearrangement-invariant spaces; to obtain a resolvent formula for the Hilbert transform in a rearrangement-invariant space; to represent generalized Cesàro operators by strongly continuous groups and semigroups; to determine the spectra of the corresponding generators; and to describe the spectra of generalized Cesàro operators using functional calculus and spectral mapping theorems.

Object of the research. The Hilbert transform and generalized Cesàro operators defined on the real line, the half-line, and a finite interval, as well as the associated weighted singular integral operators, strongly continuous groups and semigroups, and their generators.

Subject of the research. The boundedness, spectrum, point spectrum, resolvent set, explicit resolvent formulas, and the dependence of the spectrum on the Boyd indices for the above operators in rearrangement-invariant and Lorentz spaces.

Research methods. The dissertation uses the spectral theory of Banach algebras and linear operators, the theory of rearrangement-invariant spaces, submajorization methods, interpolation theory, estimates for dilation operators and Boyd indices, the theory of strongly continuous groups and semigroups, methods for studying the spectra of generators, the Hille–Phillips functional calculus and

functional calculus for sectorial operators, spectral mapping theorems, integral transforms related to the Beta and Gamma functions, analytic continuation, and comparison of boundary values of the resolvent.

Theoretical and practical significance. The dissertation is theoretical in nature. The results contribute to the development of the spectral theory of integral and singular integral operators on rearrangement-invariant spaces. The spectral formulas obtained for Lorentz spaces extend classical results for Lebesgue spaces, while the descriptions in terms of Boyd indices simultaneously cover a broad range of rearrangement-invariant spaces. The approaches developed in the dissertation can be applied to the study of other singular integral operators and Hardy-type operators. The results may also be used in advanced courses on functional analysis, harmonic analysis, operator theory, and spectral theory.

Scientific novelty and the principal results submitted for defense. The spectral properties of the Hilbert transform and generalized Cesàro operators are studied in Lorentz and rearrangement-invariant spaces within a unified methodological framework, and the following results are obtained:

- the algebraic property of the Hilbert transform on the real line in Lorentz spaces is established; its spectrum and point spectrum are shown to be equal to the set $\{-1, 1\}$, and an explicit formula for its resolvent is obtained;
- the spectrum of the Hilbert transform on the half-line in Lorentz spaces is described as a circular arc, and the resolvent is represented in terms of a logarithmic parameter and a weighted Hilbert transform;
- a resolvent formula for the Hilbert transform on the half-line is obtained, and the relationship between its spectrum and the region determined by the Boyd indices is established;
- integral representations of generalized Cesàro operators in terms of strongly continuous groups and semigroups are obtained, and their boundedness and spectra in Lorentz spaces are described by ratios of Gamma functions;
- generalized Cesàro operators on rearrangement-invariant spaces with nontrivial Boyd indices are investigated by means of functional calculus, and formulas for their spectra on the domains $(0,1)$ and $(0,\infty)$, depending on the lower and upper Boyd indices of the space, are obtained.

Reliability and validity of the scientific statements, conclusions, and results. The reliability of the results is ensured by the use of established methods of functional analysis and operator theory, rigorous proofs of all principal statements, consistency of the formulas with known special cases, and publication of the results in peer-reviewed scientific journals. For Lorentz spaces, when $q=p$, the obtained results agree with the known spectral descriptions in classical Lebesgue spaces,

while the formulas for rearrangement-invariant spaces reduce to previously known results for the classical values of the Boyd indices.

Approbation. Selected results of the dissertation were presented and discussed at the traditional International Mathematical April Conferences dedicated to the Day of Science Workers (Almaty, 2023–2025), the EMJ-2025 International Conference (Astana, 2025), the 15th ISAAC Congress (Astana, 21–25 July 2025), the international conference “Modern Trends in Fourier Analysis” held at the Centre de Recerca Matemàtica (Barcelona, 2–6 June 2025), and scientific seminars of the Institute of Mathematics and Mathematical Modeling and Al-Farabi Kazakh National University.

Publications. The principal results of the dissertation have been published in three research articles:

1. M. E. Akhymbek, R. A. Tastankul, B. O. Ozbekbay. Spectrum of the Hilbert Transform on Orlicz Spaces over R. Journal of Mathematics, Mechanics and Computer Science, 121 (2024), no. 1, 3–11. DOI: 10.26577/JMMCS202412111. Квартиль:Q4, Процентиль: 18, Индекс цит.: 1.0, SJR: 0.275.

2. M. Akhymbek, K. Tulenov, B. Ozbekbay. Spectrum of the Hilbert Transform on Lorentz Spaces Lp,q. Complex Variables and Elliptic Equations, 70 (2025), no. 9, 1555–1566. DOI: 10.1080/17476933.2024.2413126. Квартиль: Q3, Процентиль: 56, Индекс цит.: 1.9, SJR: 0.493.

3. K. Tulenov, B. Ozbekbay. Spectrum of the Generalized Cesàro Operator on the Lorentz Spaces. Journal of Mathematics, Mechanics and Computer Science, 125 (2025), no. 1, 66–80. DOI: 10.26577/JMMCS2025125106. Квартиль: Q2, Процентиль: 37, Индекс цит.: 1.0, SJR: 0.275.

Structure of the dissertation. The dissertation consists of an introduction, three chapters, a conclusion, and a list of references.

Content of the dissertation. The introduction substantiates the relevance of the research topic and formulates the aim and objectives of the work, the object and subject of the research, the methods used, the scientific novelty, and the theoretical significance.

Chapter One systematizes the theoretical framework required to prove the principal results of the dissertation. Rearrangement-invariant Banach function spaces, the distribution function, the decreasing rearrangement, Hardy–Littlewood–Pólya submajorization, the fundamental function, dilation operators, and Boyd indices are introduced. Lebesgue, Lorentz, and Orlicz spaces are considered as the principal examples of rearrangement-invariant spaces. The chapter also presents Banach algebras, algebraic operators, the minimal polynomial, the spectrum and resolvent of an operator, and the notions of point, continuous, and residual spectra.

The final sections describe the necessary results on strongly continuous groups and semigroups, their generators, growth bounds, and functional calculus for sectorial operators.

Chapter Two is devoted to the spectral properties of the Hilbert transform. First, the versions of the Hilbert transform defined on the real line and the half-line are considered, and their dependence on the function space and the domain of definition is described. In Lorentz spaces, an algebraic identity relating the square of the Hilbert transform on the real line to the identity operator is used. On the half-line, the resolvent is expressed through a parameter $w(\lambda)$, determined by a chosen branch of the logarithm, and a weighted Hilbert transform. Using interpolation properties of Lorentz spaces, the resolvent formula is extended, and the geometric locus of the spectrum is described as a circular arc joining the points -1 and 1.

Chapter Three investigates the spectrum of generalized Cesàro operators. First, membership conditions for power functions in Lorentz spaces are established, and special dilation operators are shown to form a strongly continuous group and semigroup. Their generators are written in differential form, and their point and full spectra are described. The generalized Cesàro operator is represented as an operator-valued integral with respect to the corresponding group or semigroup. This method makes it possible to prove boundedness and to describe the spectrum as the image of the generator spectrum under a ratio of Gamma functions.

In the subsequent part of the chapter, this approach is extended to rearrangement-invariant spaces with nontrivial Boyd indices. The growth properties of strongly continuous groups and semigroups are studied through the norms of dilation operators, and the spectra of their generators are determined. Generalized Cesàro operators are treated as functions of the generators by means of the Hille–Phillips functional calculus.

The conclusion summarizes the principal results of the dissertation and compares the spectral nature of the Hilbert transform and generalized Cesàro operators in Lorentz and rearrangement-invariant spaces.