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Numerical - analytical methods for solving nonlocal problems for integro-differential equations of parabolic type

ABSTRACT

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Partial differential equations of parabolic type play a fundamental role in modern mathematics and are widely used to model a variety of processes arising in science and engineering. Boundary value problems for parabolic equations are extensively applied to the analysis of heat transfer in engineering systems, groundwater flow, climate modeling, biological and chemical processes, environmental applications, materials science, and financial engineering.

Research relevance. At present, processes with memory effects are the subject of extensive research in mathematical physics, heat conduction theory, diffusion, hydrodynamics, biology, and control theory. Such processes are commonly described by integro-differential equations of parabolic type and loaded parabolic equations. A characteristic feature of these equations is that the state of the system depends not only on its current state but also on its previous and, in some cases, subsequent states. Consequently, the theory of loaded and integro-differential equations has developed rapidly and remains one of the active research areas in the modern theory of differential equations.

Significant contributions to the development of the theory of loaded parabolic equations have been made by A.D. Iskenderov, A.M. Nakhushev, M.T. Dzhenaliev, Kh.Zh. Dikinov, A.A. Kerefov, A.P. Belogurov, S.Kh. Gekkieva, M.A. Kerefov, R.I. Parovik, V.M. Borok, Ya.I. Zhitomirskii, Liu Junyi, Xu Mingyu, M.Kh. Shkhanukov, and many other researchers. In particular, A.M. Nakhushev and V.I. Borisov investigated loaded parabolic equations obtained by linearizing the classical Boussinesq equation. A.D. Iskenderov demonstrated that a number of inverse problems can be reduced to loaded parabolic equations, while N.N. Kochina showed that problems related to groundwater level regulation in irrigation systems can also be formulated in terms of equations of this class.

It is well known from mathematical analysis that quadrature formulas are widely used for the numerical approximation of definite integrals: $\int_a^b f(x)dx \approx \sum_{k=0}^{N} c_k f(x_k)$, where x_k are the quadrature nodes and c_k - are the corresponding quadrature weights. In numerical analysis, this approach is commonly employed to reduce integro-differential equations to loaded differential equations. In particular, A.M. Nakhushev considered integro-differential equations of parabolic type of the form $\frac{\partial u(t,x)}{\partial t} = \Delta_x u(t,x) + \int_0^T k(t,s)u(s,x)ds$, where Δ_x denotes the Laplace operator, and $k(t,s)$ - is the kernel of the integral operator describing the space-time dependence of the

integral term. Let the interval $0 \le s \le T$ be partitioned as $0 \le t_0 < t_1 < \ldots < t_m < t_{m+1} = T$. Assume that the kernel is piecewise constant, i.e., $k(t,s) = a_i(t)$, $t_i \le s \le t_{i+1}$, $i = 0,1,\ldots,m$. Then the integral term can be represented as $\int_0^T k(t,s)u(s,x)ds = \sum_{i=0}^{m} a_i(t) \int_{t_i}^{t_{i+1}} u(s,x)ds$. Applying a quadrature formula yields the approximation $\int_0^T k(t,s)u(s,x)ds \approx \sum_{j=0}^{m} b_j(t)u(t_j,x)$, where $b_j(t)$ - are the quadrature weights. Thus, this transformation provides a theoretical basis for reducing mathematical models governed by integro-differential equations of parabolic type to loaded parabolic equations. Therefore, loaded parabolic equations can be regarded as either a special case or an approximate model of Fredholm integro-differential equations of parabolic type.

Research objective. The objective of this dissertation is to establish conditions ensuring the unique solvability of nonlocal problems for loaded parabolic equations and integro-differential equations of parabolic type, as well as to develop analytical and numerical-analytical methods for solving these problems.

Research object. The object of the research is nonlocal problems for loaded parabolic equations and Fredholm integro-differential equations of parabolic type, together with numerical-analytical methods for their solution.

Research methods. The theoretical framework of this dissertation is based on the theory of differential and integro-differential equations, the theory of nonlocal problems for partial differential equations, the method of lines, the Dzhumabaev parameterization method, and methods of numerical analysis.

Theoretical and practical significance of the research. This dissertation investigates the solvability of nonlocal boundary value problems for loaded parabolic equations and Fredholm integro-differential equations of parabolic type. The obtained results contribute to the development of the theory of loaded and integro-differential equations. The proposed methods can be applied to the study of nonlocal boundary value problems, integro-differential equations, and loaded equations.

Scientific novelty of the research, main results submitted for defense. The scientific novelty of the dissertation is reflected in the following results:

- the method of lines has been developed and justified for solving nonlocal problems for loaded parabolic equations and Fredholm integro-differential equations of parabolic type, and the corresponding discretized problems have been derived;

-the Dzhumabaev parameterization method has been applied to the discretized systems of loaded differential equations and Fredholm integro-differential equations, leading to the development of algorithms for solving the corresponding boundary value problems;

-conditions ensuring the unique solvability of the investigated problems have been established, and the feasibility of constructing their numerical-analytical solutions has been demonstrated;

-conditions guaranteeing the convergence, stability, and accuracy of the method of lines for solving nonlocal boundary value problems for loaded parabolic equations and Fredholm integro-differential equations of parabolic type have been established;

-the proposed methods and the obtained results contribute to the further development of the theory of nonlocal boundary value problems and can be applied to the mathematical modeling of complex processes in mathematical physics and applied mathematics.

Approbation of the work. The main results of the dissertation have been presented at international and national scientific conferences, as well as at scientific seminars. A total of 16 scientific publications have been published on the topic of the dissertation.

Publications indexed in the Web of Science and Scopus databases:

1. **Kuanysh S.K.,** Assanova A.T., Kadirbayeva Zh.M. *Method of Lines for a Loaded Parabolic Equation.* Journal of Mathematics, Mechanics, Computer Science, 1(125), 2025, pp. 3–17. *Scopus* (CiteScore Percentile -37%), *Web of Science* (Q3).
2. Assanova A.T., Kadirbayeva Zh.M., **Kuanysh S.K.** *Numerical-Analytical Method for Solving an Initial-Boundary Value Problem for a Loaded Parabolic Equation.* Bulletin of the Karaganda University. Mathematics Series, 1(117), 2025, pp. 34–45. *Scopus* (CiteScore Percentile -53%), *Web of Science* (Q2).
3. Assanova A.T., Liu Zh., **Kuanysh S.K.,** Kadirbayeva Zh.M. *Convergence, Stability, and Error Analysis of the Method of Lines for Solving a Loaded Parabolic Equation.* AIMS Mathematics, 10(12), 2025, pp. 29454–29469. *Scopus* (CiteScore Percentile -86%), *Web of Science* (Q1).
4. **Kuanysh S.K.,** Kadirbayeva Zh.M. *Analysis of a Nonlocal Problem for a Loaded Parabolic Equation via a Numerical Method.* Lobachevskii Journal of Mathematics, 46(10), 2025, pp. 5417-5427. *Scopus* (CiteScore Percentile -56%), *Web of Science* (Q2).
5. Temesheva S.M., Kadirbayeva Zh.M., **Kuanysh S.K**. *Analysis of a Semi-Periodic Problem for Integro-Differential Equations of Parabolic Type.* Lobachevskii Journal of Mathematics, 46(12), 2025, pp. 6441-6454. *Scopus* (CiteScore Percentile -56%), *Web of Science* (Q2).

Publication in a National Scientific Journal:

6. Iskakova N.B., Kadirbayeva Zh.M., Bakirova E.A., **Kuanysh S.K.** *Numerical method for solving the boundary value problem for the parabolic equation. Herald of the Kazakh-British Technical University*, Vol. 21, No. 3, 2024, pp. 165-175.

Publications in the Proceedings of International Scientific Conferences:

1. **Kuanysh S.K.** *On a Method for Solving a Nonlocal Boundary Value Problem for a Parabolic Equation* // International Scientific Conference of Students and Young Scientists "Farabi Alemi". - Almaty: Al-Farabi Kazakh National University, April 4–6, 2024. -P. 26.
2. **Kuanysh S.** *Algorithm for Solving the Two-Point Boundary Value Problem for Loaded Parabolic Equations // Proceedings of the CITech–2024 Conference "Computational and Information Technologies in Science, Engineering and Education".* -Almaty, Kazakhstan, October 23–24, 2024. -P. 93.
3. **Kuanysh S.** *Solution of Nonlocal Problem for Loaded Parabolic Equations // Traditional International April Mathematical Conference in Honor of the Kazakhstan Day of Science Workers.* -Almaty, Kazakhstan, April 1-4, 2025. -P. 158.
4. **Kuanysh S.K.** *A Two-Point Boundary Value Problem for Loaded Parabolic Equations* // International Scientific Conference of Students and Young Scientists "Farabi Alemi" -Almaty: Al-Farabi Kazakh National University, April 3–5, 2025. -P. 4.
5. **Kuanysh S.** *Numerical Solution of a Periodic Problem for Integro-Differential Equations of Parabolic Type* // International Scientific and Practical Conference "Digital Technologies and Interdisciplinary Approaches to Solving Applied Problems in Education and Science". -Atyrau, Kazakhstan, October 3, 2025. -P. 342.
6. **Kuanysh S.K.,** Bakirova E.A. *A Stable and Convergent Method of Lines for Fredholm-Type Parabolic Integro-Differential Equations // The First Congress of Kazakhstan Mathematicians.* -Shymkent, Kazakhstan, April 3–7, 2025. -P. 238.
7. **Kuanysh S.K.,** Iskakova N.B., Abdimanapova P.B. *Convergence, Stability, and Error Analysis of the Method of Lines for Solving Loaded Parabolic Equations // The First Congress of Kazakhstan Mathematicians.* -Shymkent, Kazakhstan, April 3–7, 2026. -P. 239.
8. **Kuanysh S.K.,** Kuanysh Zh.K. *Classical Solutions of Parabolic Integro-Differential Equations with Nonlocal-in-Time Conditions* // International Scientific Conference of Students and Young Scientists "Farabi Alemi"- Almaty: Al-Farabi Kazakh National University, April 7–8, 2026.
9. **Kuanysh S.** *Numerical Analysis of Parabolic Volterra Integro-Differential Equations // Fundamental and Applied issues in Mathematics, Mechanics, and Computer Science.* -Karaganda: Buketov Karaganda National Research University, June 3–4, 2026. -P. 61.
10. **Kuanysh S.K.** *Classical Solutions of Parabolic Integro-Differential Equations with Nonlocal-in-Time Boundary Conditions // Integral Operators in Function Spaces and Harmonic Analysis International Scientific Conference.* -Astana: L.N. Gumilyov Eurasian National University, June 11–14, 2026. -P. 148.

In addition, the results of the dissertation were presented at the 14th AIMS International Conference on Dynamical Systems, Differential Equations and Applications (Abu Dhabi, UAE, December 16–20, 2024), the 10th International Scientific Conference "Mathematics, Mechanics and Information Technologies" (Almaty, Kazakhstan, September 24–26, 2025), and the International CMMSE Conference and the Fourth Conference on High Performance Computing (CHPC) (Cádiz, Spain, July 2–9, 2026). No conference proceedings were published for these events.

Structure of the dissertation. The dissertation consists of an introduction, two chapters, a conclusion, and a list of references. **Chapter 1** is devoted to the study of nonlocal boundary value problems for loaded parabolic equations. **Section 1.1** presents the formulation of a nonlocal boundary value problem for a loaded parabolic equation. By applying the method of lines, the corresponding discretized problem is derived. **Section 1.2** applies the classical Dzhumabaev parameterization method to the discretized system of loaded differential equations. Conditions ensuring unique solvability are established, and an algorithm for constructing an approximate solution is proposed. **Section 1.3** investigates the convergence, stability, and accuracy of the method of lines applied to the considered boundary value problems. **Section 1.4** presents numerical examples and computational results for loaded parabolic equations. **Section 1.5** formulates the initial-boundary value problem for loaded parabolic equations and analyzes the corresponding solution methods. **Section 1.6** establishes the unique solvability of the initial-boundary value problem for loaded parabolic equations. **Section 1.7** considers the application of the Dzhumabaev parameterization method to the discretized problem and establishes conditions for its unique solvability. **Section 1.8** presents an algorithm for the numerical solution of the initial-boundary value problem for loaded parabolic equations together with numerical examples. **Chapter 2** is devoted to the study of nonlocal boundary value problems for Fredholm integro-differential equations of parabolic type. **Section 2.1** presents the formulation of a nonlocal boundary value problem for Fredholm integro-differential equations of parabolic type. By applying the method of lines, a discretized nonlocal boundary value problem for a system of Fredholm integro-differential equations is derived. **Section 2.2** applies the Dzhumabaev parameterization method to the resulting discretized nonlocal problem and develops the corresponding solution algorithm. **Section 2.3** investigates the convergence, stability, and accuracy of the method of lines for solving nonlocal boundary value problems for Fredholm integro-differential equations of parabolic type. **Section 2.4** presents numerical examples for Fredholm integro-differential equations of parabolic type together with computational results demonstrating the accuracy of the proposed method and providing estimates of the numerical error. **The Conclusion** summarizes the principal scientific results obtained in the dissertation and presents the main conclusions.