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Dunkl analysis and application to inverse source problems

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Summary

In our thesis we consider:

- pseudo-differential operators generated by the Dunkl operator;
- inverse source problem for Dunkl-heat equation with the Caputo fractional derivative;
- inverse source problem for Dunkl-pseudo-parabolic equation with the Caputo fractional derivative;
- inverse source problem for Dunkl-heat equation with the bi-ordinal Hilfer fractional derivative.

In the first chapter, we collect some basic results in the Dunkl analysis and fractional analysis. We define the Dunkl operator D_α and the Dunkl Laplacian D_α^2 on the suitable spaces and consider properties of the Dunkl operator. We define the Dunkl kernel $E_\alpha(x, y)$ as a unique solution of the initial value problem generated by the Dunkl operator. Then we obtain series and Poisson integral representations of the Dunkl kernel. We prove that the Dunkl kernel $E_\alpha(x, y)$ does not have zeros for all $x, y \in \mathbb{R}$. Then we define Dunkl and inverse Dunkl transforms on $L^1(\mathbb{R}, d\mu_\alpha)$ and study their properties. After we define the Dunkl transform on tempered distributions $\mathcal{S}'(\mathbb{R})$ and prove that it is a continuous linear transformation on $\mathcal{S}'(\mathbb{R})$. Also, we give Taylor series generated by the Dunkl operator, as a part of Dunkl analysis.

In the second chapter, we consider pseudo-differential operators generated by the Dunkl operator. Some boundedness results for these operators were already known in the literature. We define amplitude, adjoint and transpose operators and prove that pseudo-differential, amplitude, adjoint and transpose operators are linear transformations on the Schwartz spaces. We also define pseudo-differential operators on tempered distributions $\mathcal{S}'(\mathbb{R})$ and prove that it is a continuous linear transformation on $\mathcal{S}'(\mathbb{R})$. Then we study properties of the distributional and convolution kernels of the pseudo-differential operators. In particular, we prove Schur's lemma. We obtain some boundedness results on spaces $L(\mathbb{R}, d\mu_\alpha)$ for the pseudo-differential operators and composition of the pseudo-differential operators, under certain assumptions.

In the last chapter, we study inverse source problems for Dunkl-heat and Dunkl-pseudo-parabolic equations with Caputo and bi-ordinal Hilfer fractional differential operators. For this inverse source problems, we prove well-posedness results in the sense of Hadamard. First, we consider direct problems and establish the unique existence of a generalized solution. Then we consider inverse source problems and define pair of solutions in suitable spaces. We use classical Fourier method. After we establish stability results, which means that the solution of the inverse source problems continuously depends on given data. Additionally, we consider some examples to give an illustration of our analysis.

Samenvatting

In onze thesis beschouwen we:

- pseudo-differentiaaloperatoren gegenereerd door de Dunkl-operator;
- inverse bronprobleem voor de Dunkl-warmtevergelijking met de Caputo fractionele afgeleide;
- inverse bron probleem voor Dunkl-pseudo-parabolische vergelijking met de Caputo fractionele afgeleide;
- inverse bron probleem voor Dunkl-warmtevergelijking met de bi-ordinale Hilfer fractionele afgeleide.

In het eerste hoofdstuk verzamelen we enkele basisresultaten in de Dunkl analyse en fractionele analyse. We definiëren de Dunkl operator D_α en de Dunkl Laplaciaan D_α^2 op de geschikte ruimtes en beschouwen eigenschappen van de Dunkl operator. We definiëren de Dunkl kern $E_\alpha(x, y)$ als een unieke oplossing van het beginwaardeprobleem gegenereerd door de Dunkl operator. Dan krijgen we reeksen en Poisson integraalrepresentaties van de Dunkl kern. We bewijzen dat de Dunkl kern $E_\alpha(x, y)$ geen nulpunten heeft voor alle $x, y \in \mathbb{R}$. Dan definiëren we de Dunkl en inverse Dunkl transformaties op $L^1(\mathbb{R}, d\mu_\alpha)$ en bestuderen hun eigenschappen. Daarna definiëren we de Dunkl transformatie op getemperde distributies $\mathcal{S}'(\mathbb{R})$ en bewijzen dat het een continue lineaire transformatie is op $\mathcal{S}'(\mathbb{R})$. Ook geven we Taylor-reeksen gegenereerd door de Dunkl operator, als onderdeel van Dunkl analyse.

In het tweede hoofdstuk beschouwen we pseudo-differentiaaloperatoren die door de Dunkl operator worden gegenereerd. Sommige resultaten van deze operatoren waren al bekend in de literatuur. We definiëren amplitude-, toegevoegde en getransponeerde operatoren en bewijzen dat pseudo-differentiële, amplitude-, toegevoegde en getransponeerde operatoren lineaire transformaties zijn op de Schwartz-ruimtes. We definiëren ook pseudo-differentiaaloperatoren op getemperde distributies $\mathcal{S}'(\mathbb{R})$ en bewijzen dat het een continue lineaire transformatie is op $\mathcal{S}'(\mathbb{R})$. Vervolgens bestuderen we de eigenschappen van de distributie- en convolutie-kernen van de pseudo-differentiaaloperatoren. In het bijzonder bewijzen we Schur's lemma. We verkrijgen enkele begrensdheidsresultaten op de ruimten $L(\mathbb{R}, d\mu_\alpha)$ voor de pseudo-differentiaaloperatoren en samenstelling van de pseudo-differentiaaloperatoren, onder bepaalde veronderstellingen.

In het laatste hoofdstuk bestuderen we inverse bronproblemen voor Dunkl-warmte- en Dunkl-pseudo-parabolische vergelijkingen met Caputo en bi-ordinale Hilfer fractionele differentiaaloperatoren. Voor deze inverse bronproblemen bewijzen we welgesteldheidsresultaten in de zin van Hadamard. Ten eerste beschouwen we directe problemen en stellen we het unieke bestaan van een veralgemeende oplossing vast. Dan beschouwen we inverse bronproblemen en definiëren we een paar oplossingen in geschikte ruimtes. We gebruiken de klassieke Fourier methode. Daarna verkrijgen we stabiliteitsresultaten, wat betekent dat de oplossing van de inverse bronproblemen continu afhankelijk is van de gegevens. Daarnaast beschouwen we enkele voorbeelden om een illustratie te geven van onze analyse.

List of symbols

1. $\mathbb{N} := \{0, 1, 2, ...\}$ is the set of natural numbers;
2. $\mathbb{Z}^+ := \{1, 2, 3, ...\}$ is the set of all positive integers;
3. $\mathbb{R}$ is the set of all real numbers;
4. $\mathbb{R}^+$ is the set of all positive real numbers.
5. $\mathbb{C}$ is the set of all complex numbers;
6. $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$;
7. $S^m(\mathbb{R} \times \mathbb{R})$ denotes $S^m_{1,0}(\mathbb{R} \times \mathbb{R})$;
8. C^k_n denotes $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Inhoudsopgave

1. Introduction

The thesis deals with the Dunkl analysis. The Dunkl analysis starts from C.F. Dunkl's works [28, 29, 30, 31, 32]. The Dunkl operators are differential-difference operators associated with some finite reflection groups. The motivation for this subject is that the Dunkl operators play an important role in the study of special functions with reflection symmetries. In Dunkl analysis, we are interested in studying pseudo-differential operators generated by the Dunkl operators on the real line and the application of Dunkl analysis in the theory of inverse source problems. The motivation for studying pseudo-differential operators generated by the Dunkl operators is to provide a necessary tool for studying partial differential equations generated by the Dunkl operators. We focus on studying pseudo-differential operators generated by the Dunkl operators on the real line because it was originally built on the real line, and we aim to extend it.

We start our thesis with an outline of the general concepts (Chapter 2): the Dunkl operator, the Dunkl kernel, the Dunkl transform, and the Dunkl convolution. Moreover, some fundamental definitions from function analysis and fractional calculus are also presented here.

Chapter 3 of our thesis is dedicated to the pseudo-differential operators generated by the Dunkl operators on the real line, and we work with Dunkl analysis on the real line. This analysis was firstly introduced by A. Dachraoui in 2001 [24]. In [24] author, after carefully revising the Harmonic analysis associated with the Dunkl operators, defined two class of symbols S_0^m and S^m, $m \in \mathbb{R}$, (definitions are given below) with $S^m \subset S_0^m$ and proved that pseudo-differential operator T_a is continuous operator from $\mathcal{S}(\mathbb{R})$ into itself for $a \in S_0^m$, where $\mathcal{S}(\mathbb{R})$ is usual Schwartz space. Two class of symbols S_0^m and S^m and the operator T_a are defined via following definitions [24].

Definition 1.1. Let $m \in \mathbb{R}$. The function $a : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S_0^m, if it satisfies

- For fixed x in $\mathbb{R}$, the function $\lambda \mapsto a(x, \lambda)$ is smooth function on $\mathbb{R}$;
- For fixed λ in $\mathbb{R}$, the function $x \mapsto a(x, \lambda)$ is smooth function on $\mathbb{R}$;
- For all $k, n \in \mathbb{N}$, there exists $C_{k,n,m} > 0$, such that

$$\left| \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} a(x, \lambda) \right| \leq C_{k,n,m} (1 + |\lambda|^2)^{\frac{m-n}{2}}$$

  for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Definition 1.2. Let $m \in \mathbb{R}$. The function $a : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is called a symbol in the class S^m, if it satisfies

- For fixed x in $\mathbb{R}$, the function $\lambda \mapsto a(x, \lambda)$ is smooth function on $\mathbb{R}$;
- For fixed λ in $\mathbb{R}$, the function $x \mapsto a(x, \lambda)$ is smooth function on $\mathbb{R}$;
- For all $k, \ell, n \in \mathbb{N}$, there exists $C_{k,\ell,n,m} > 0$, such that

$$\left| (1 + |x|^2)^\ell \frac{\partial^k}{\partial x^k} \frac{\partial^n}{\partial \lambda^n} a(x, \lambda) \right| \leq C_{k,\ell,n,m} (1 + |\lambda|^2)^{\frac{m-n}{2}}$$

  for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Definition 1.3. Let $a \in S_0^m$ and $\alpha \geq -1/2$. The pseudo-differential operator associated with a symbol a is defined on $\mathcal{S}(\mathbb{R})$ by

$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda),$$

where E_α is the Dunkl kernel defined by

$$E_\alpha(x,\lambda) = j_\alpha(x\lambda) + i\frac{x\lambda}{2(\alpha+1)} j_{\alpha+1}(x\lambda),$$

j_α is the normalized Bessel function of the first kind, $\mathcal{F}_\alpha[f]$ is the Dunkl transform given by

$$\mathcal{F}_\alpha[f](\lambda) = \int_{\mathbb{R}} E_\alpha(-x,\lambda) f(x) d\mu_\alpha(x),$$

and

$$d\mu_\alpha(\lambda) = \frac{|\lambda|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} d\lambda,$$

Γ is a Gamma function.

Also in [24], author introduced Sobolev type spaces $W_\alpha^{s,p}(\mathbb{R}, d\mu_\alpha)$, $s \in \mathbb{R}$, $p \in [1,+\infty]$, (next definition) and proved that the operator T_a with symbol $a \in S^m$, is continuous from $W_\alpha^{\frac{m}{2},1}(\mathbb{R}, d\mu_\alpha)$ into $W_\alpha^{0,\infty}(\mathbb{R}, d\mu_\alpha)$, and from $W_\alpha^{\frac{m}{2},p}(\mathbb{R}, d\mu_\alpha)$ into $W_\alpha^{0,p}$, $p \geq 1$.

Definition 1.4. The space $W_\alpha^{s,p}(\mathbb{R}, d\mu_\alpha)$, $s \in \mathbb{R}$, $p \in [1,+\infty]$, is defined as the closure of a space of C^∞-functions on $\mathbb{R}$ with compact support, with respect to the norms

$$\|f\|_{W_\alpha^{s,p}} := \|(1+\lambda^2)^{s/2} \mathcal{F}_\alpha[f]\|_{p,\alpha}, \quad \text{if} \quad p \in [1,+\infty),$$

and

$$\|f\|_{W_\alpha^{s,\infty}} := \sup_{\lambda\in\mathbb{R}} (1+\lambda^2)^{s/2} |\mathcal{F}_\alpha[f](\lambda)| \quad \text{if} \quad p = +\infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}}.$$

We proved that the pseudo-differential operator T_a is a continuous linear operator on $\mathcal{S}(\mathbb{R})$ for $a \in S_{\rho,\delta}^m(\mathbb{R}\times\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, where $S_{\rho,\delta}^m(\mathbb{R}\times\mathbb{R})$ is a classical symbol class defined by the following definition.

Definition 1.5 (Symbol classes $S_{\rho,\delta}^m(\mathbb{R}\times\mathbb{R})$). Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. If $a = a(x,\lambda)$ is in $C^\infty(\mathbb{R}\times\mathbb{R})$ and

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \leq C_{n,k}(1+|\lambda|)^{m-\rho n+\delta k}$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$. Then we will say that $a \in S_{\rho,\delta}^m(\mathbb{R}\times\mathbb{R})$.

After, same continuity results follows for amplitude, adjoint and transpose operators defined on $\mathcal{S}(\mathbb{R})$. Definitions of these operators can be found in Section 3.1. Additionally, we obtained some boundedness results of pseudo-differential operators generated by the Dunkl operators under certain assumptions listed below.

Definition 1.6. Let us define the space $L(\mathbb{R}, d\mu_\alpha)$, as following

$$L(\mathbb{R}, d\mu_\alpha) := \{f \in L^1(\mathbb{R}, d\mu_\alpha) \ : \ \mathcal{F}_\alpha[f] \in L^1(\mathbb{R}, d\mu_\alpha)\}$$

with a norm

$$\|f\|_L := \|\mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where $L^1(\mathbb{R}, d\mu_\alpha)$ is a space of (Lebesgue) measurable functions on $\mathbb{R}$ with the norm $\|\cdot\|_{1,\alpha}$.

Assumption 1.7. *We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$ is defined as:*

$$a(x, \lambda) = \int_{\mathbb{R}} E_\alpha(x, \xi) V(\xi, \lambda) d\mu_\alpha(\xi),$$

where $V(\xi, \lambda)$ is a complex valued measurable function on $\mathbb{R} \times \mathbb{R}$, such that

$$|V(\xi, \lambda)| \leq K(\xi),$$

for all $\xi, \lambda \in \mathbb{R}$ and $K \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Theorem 1.8. *Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator T_a is a bounded linear operator under Assumption 1.7 on $L(\mathbb{R}, d\mu_\alpha)$, i.e.*

$$\|T_a f\|_L \leq 4\|K\|_{1,\alpha}\|f\|_L.$$

Corollary 1.9. *Let T_a and T_b are pseudo-differential operators with symbols a and b, respectively. Then under Assumption 1.7 their composition is a pseudo differential operator $T_a \circ T_b$, which is continuous linear map on $\mathcal{S}(\mathbb{R})$.*

Corollary 1.10. *Let $f \in \mathcal{S}(\mathbb{R})$. Then under Assumption 1.7 the composition of pseudo-differential operator T_a and T_b is a bounded linear operator on $L(\mathbb{R}, d\mu_\alpha)$, i.e.*

$$\|T_a(T_b f)\|_L \leq \frac{16}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha}\|f\|_L.$$

Assumption 1.11. *We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$ is defined by*

$$a(x, \lambda) = \int_{\mathbb{R}} E_\alpha(x, \xi) V(\xi, \lambda) d\mu_\alpha(\xi),$$

satisfies

$$a(x, \lambda) = \int_{\mathbb{R}} E_\alpha(x, \xi) V_1(\xi) V_2(\lambda) d\mu_\alpha(\xi) = V_2(\lambda) \int_{\mathbb{R}} E_\alpha(x, \xi) V_1(\xi) d\mu_\alpha(\xi),$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Theorem 1.12. *Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator T_a with symbol $a(x, \lambda)$, which satisfies Assumption 1.11, has a representation*

$$T_a f(x) = 2^{\alpha+1}\Gamma(\alpha+1)\mathcal{F}_\alpha^{-1}(V_1 *_\alpha V_2 \mathcal{F}_\alpha[f])(x)$$

and satisfies following inequality

$$\|T_a f\|_L \leq 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2 \mathcal{F}_\alpha[f]\|_{1,\alpha}.$$

Assumption 1.13. *We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is defined by*

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V(\xi,\lambda)d\mu_\alpha(\xi),$$

satisfies

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)V_2(\lambda)d\mu_\alpha(\xi) = V_2(\lambda)\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi),$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function and $V_2(\lambda) = A$ is a constant. So we have

$$a(x,\lambda) = A\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi).$$

Theorem 1.14. *Let $f \in \mathcal{S}(\mathbb{R})$. Then the composition of the pseudo-differential operators T_a and T_b with symbols a and b, which satisfy Assumption 1.13, has a representation*

$$T_a(T_bf)(x) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 A\cdot\mathcal{F}_\alpha^{-1}[V_1 *_\alpha (W_1 *_\alpha B\cdot\mathcal{F}_\alpha[f])](x)$$

and satisfies following inequality

$$\|T_a(T_bf)\|_L \leq 16\left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\|V_1\|_{1,\alpha}\|W_1\|_{1,\alpha}\|f\|_L.$$

In classical harmonic analysis for many different classes of symbols were studied boundedness properties of the pseudo-differential operators. As well, many results were obtained and extended to the pseudo-differential operators associated with the Dunkl operator. In [1], L^2 and L^p-boundedness of the pseudo-differential operator T_a associated with the Dunkl operator was studied by the authors C. Abdelkeffi, B. Amri, and M. Sifi for class of symbols $S^0_{1,0}$, or simply S^0, which contains symbols with property

$$\left|\partial^n_\lambda\partial^k_x a(x,\lambda)\right| \leq C_{k,n}(1+|\lambda|)^{-n}$$

for all $k,n\in\mathbb{N}$ and $x,\lambda\in\mathbb{R}$. Also, in [1] was obtained a singular integral representation of the operator T_a, proved that the kernel of the operator T_a satisfies the condition of the singular integral theorem and defined kernel of the adjoint operator T_a^* to the operator T_a. The main results of the work [1] are as follows.

Proposition 1.15. *Assume that $a \in S^0$. Then there exists a continuous function k_a on $\mathbb{R}\times\mathbb{R}^*$ such that*

$$|k_a(x,z)| \leq \frac{C_N}{|z|^N}, \quad x,z\in\mathbb{R}, \quad z\neq 0$$

for all $N\in\mathbb{N}$, $N > 2(\alpha+1)$ and we have

$$T_af(x) = \int_{\mathbb{R}} K_a(x,y)f(y)d\mu_\alpha(y),$$

for all $f\in\mathcal{S}(\mathbb{R})$, such that the complement of $supp(f)$ is nonempty and $|x|\notin supp(f)$, where

$$K_a(x,y) = \int_{\mathbb{R}} k_a(x,-z)d\nu_{x,-y}(z)$$

given on $\{(x,y)\in\mathbb{R}^2 : |x|\neq|y|\}$. Here C_N is a constant which depends only on N and measures $d\nu_{x,y}$ are defined in Theorem 2.54.

Proposition 1.16. *Let $a \in S^0$ and T_a^* be the adjoint operator of T_a. Then we obtain*

$$T_a^* g(y) = \int_{\mathbb{R}} K_a^*(y,x) g(x) d\mu_\alpha(x)$$

where $K_a^*(y,x) = \overline{K_a(x,y)}$, for all $g \in \mathcal{S}(\mathbb{R})$, such that the complement of $supp(g)$ is nonempty and $y \in \mathbb{R}$, $|y| \notin supp(g)$.

Proposition 1.17. *Suppose that $a \in S^0$. Then T_a can be extended to a bounded operator on $L^2(\mathbb{R}, d\mu_\alpha)$.*

Theorem 1.18. *Let $a \in S^0$. Then T_a can be extended to a bounded operator on $L^p(\mathbb{R}, d\mu_\alpha)$, where $1 < p < +\infty$.*

Another work in this direction is [5] by B. Amri, S. Mustapha, and M. Sifi. In [5], authors have extended L^2-theorem of Calderón-Vaillancourt to the pseudo-differential operator T_a associated with the Dunkl operator, on the real line.

Theorem 1.19 (Calderón-Vaillancourt)**.** *Assume that $0 \le \rho < 1$ and $a \in S^0_{\rho,\rho}$, which is $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ and satisfies*

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \le C_{n,k}(1+|\lambda|)^{\rho(k-n)}$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$. Then T_a can be extended to a bounded operator on $L^2(\mathbb{R}, d\mu_\alpha)$.

Also, in [5] was obtained L^p-boundedness of the operator T_a with symbols in $S^0_{1,\delta}$, $0 \le \delta < 1$. We say that a symbol a belongs to the class $S^0_{1,\delta}$, $0 \le \delta < 1$ if $a \in C^\infty(\mathbb{R} \times \mathbb{R})$ and satisfies

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \le C_{n,k}(1+|\lambda|)^{-n+\delta k}$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$.

Theorem 1.20. *Let $a \in S^0_{1,\delta}$, $0 \le \delta < 1$. Then T_a can be extended to a bounded operator from $L^p(\mathbb{R}, d\mu_\alpha)$ into itself, for all $1 < p < +\infty$.*

In thesis, we obtained following kernel theorems:

Theorem 1.21 (Kernel of a pseudo-differential operator). *Let $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$. Then $K(x,y)$ is C^∞ on $\{(x,y) \in \mathbb{R}^2 : |x| \neq |y|\}$, and*

$$|K(x,y)| \le \frac{C_{N,\alpha}}{||x|-|y||^N}$$

for all $N \in \mathbb{N}$ and $|x| \neq |y|$.

Theorem 1.22 (Convolution kernel of a pseudo-differential operator). *Assume that $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$. Then convolution kernel*

$$k(x,z) = \int_{\mathbb{R}} E_\alpha(z,\lambda)(1+\lambda^2)^{-\ell} a(x,\lambda) d\mu_\alpha(\lambda)$$

of the pseudo-differential operator T_a satisfies

$$|\partial_x^s k(x,z)| \le \frac{C_{n,s}}{|z|^n}, \quad x, z \in \mathbb{R}, \quad \textit{and} \quad z \neq 0$$

for $m + \delta s + 2(\alpha+1) < \ell + \rho n$.

In [1], authors have obtained this Theorem (Proposition 1.15) for the class of symbols S^0, with $m=0$, $\rho=1$, $\delta=0$, in which case we have

$$|k(x,z)| \leq \frac{C_n}{|z|^n}, \quad x, z \in \mathbb{R}, \quad \text{and} \quad z \neq 0$$

for $2(\alpha+1) < \ell+n$. So, in Theorem 1.22 we are simplifying the condition

$$2(\alpha+1) < N, \quad N \in \mathbb{N}$$

and generalizing Proposition 1.15.

Chapter 4 of our thesis is dedicated to several types of inverse source problems generated by the Dunkl operator on the real line. The Cauchy problem for the heat equation associated with the Dunkl operator

$$\begin{cases} D^2_{\alpha,x}u(t,x) - u_t(t,x) = 0 \\ u(0,t) = g(x) \end{cases}$$

was considered by M. Rösler [76] (originally work was done in $\mathbb{R}^n$) on a domain $(0,\infty)\times\mathbb{R}$ with initial data $g \in C_b(\mathbb{R})$, where partial derivatives and the usual exponential kernel are replaced by the Dunkl operators and the generalized exponential kernel of the Dunkl transform. Here the Dunkl Laplacian $D^2_{\alpha,x}$ is defined by

$$D^2_\alpha f(x) = \frac{d^2}{dx^2}f(x) + \frac{2\alpha+1}{x}\frac{d}{dx}f(x) - \left(\alpha+\frac{1}{2}\right)\frac{f(x)-f(-x)}{x^2}$$

for every $\alpha \geq -\frac{1}{2}$. Then nonhomogeneous problem

$$\begin{cases} u_t(t,x) - D^2_{\alpha,x}u(t,x) = f(t,x) \\ u(0,t) = g(x) \end{cases}$$

was considered by H. Mejjaoli [56, 57] (also originally work was done in $\mathbb{R}^n$) on a domain $(0,\infty)\times\mathbb{R}$ when g belongs to homogeneous and nonhomogeneous Dunkl–Besov spaces.

Here we considered inverse source problems for heat, and pseudo-parabolic equations with Caputo fractional derivatives, and heat equation with the bi-ordinal Hilfer fractional derivative, generated by the Dunkl operator.

In Section 4.1, we study direct and inverse source problems for heat equation generated by the Dunkl operator. First, we consider the Cauchy problem

$$\begin{cases} \mathcal{D}^\gamma_{0+,t}u(t,x) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(t,x), \quad (t,x) \in Q_T, \\ u(0,x) = g(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $Q_T := \{(t,x) : 0 < t < T < +\infty, x \in \mathbb{R}\}$, $0 < \gamma \leq 1$, m, T are given positive numbers, g is suitable given function and $\mathcal{D}^\gamma_{0+,t}$ is the left-sided Caputo fractional derivative (Definition 2.68).

A generalised solution of the Cauchy problem (1.1) is the function

$$u \in C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$$

satisfying the above equation.

Theorem 1.23. *Let $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$, $f \in C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha))$. Then there exists a unique generalised solution of the Cauchy problem (1.1). Moreover, it is given by the expression*

$$\begin{aligned} u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(y)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right) E_\alpha(x,\lambda)E_\alpha(-y,\lambda) \\ \times d\mu_\alpha(y)d\mu_\alpha(\lambda) \\ + \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t f(\tau,y)(t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-(m+\lambda^2)(t-\tau)^\gamma\right) \\ \times E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\tau d\mu_\alpha(y)d\mu_\alpha(\lambda), \end{aligned}$$

where E_α is the Dunkl kernel and $\mathbb{E}_{\gamma,1}$ and $\mathbb{E}_{\gamma,\gamma}$ are Mittag-Leffler functions.

Then we studied following inverse source problem

$$\begin{cases} \mathcal{D}^\gamma_{0+,t}u(t,x) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), & (t,x) \in Q_T, \\ u(0,x) = \phi(x), & x \in \mathbb{R}, \\ u(T,x) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\phi(x)$ and $\psi(x)$ are given suitable functions. Our aim is to find pair of functions (u, f).

A generalised solution of Inverse source problem (1.2) is a pair of functions (u, f), where

$$u \in C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$$

and $f \in L^2(\mathbb{R}, d\mu_\alpha)$.

Theorem 1.24. *Let $\psi, \phi \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then a generalised solution of Inverse source problem (1.2) exists and is unique. Moreover, it can be written by the expressions*

$$\begin{aligned} u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Bigg(\phi(y) + \frac{\psi(y)-\phi(y)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \\ \times \left(1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)\right)\Bigg) E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda) \end{aligned}$$

and

$$\begin{aligned} f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} (m+\lambda^2)\frac{\psi(y)-\phi(y)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \\ \times E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda), \end{aligned}$$

where $\mathbb{E}_{\gamma,1}$ is the classical Mittag-Leffler function.

In Section 4.2, we study direct and inverse source problems for pseudo-parabolic equation generated by the Dunkl operator, as generalizations of previous problems, given in Section 4.1. First, we considered the Cauchy problem for the the time-fractional pseudo-parabolic equation

$$\begin{cases} \mathcal{D}^\gamma_{0+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(t,x), & (t,x) \in Q_T \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $a, m > 0$, $0 < \gamma \leq 1$ and g is a given suitable function.

The following theorem shows that the Cauchy problem (1.3) has a unique generalised solution in the space $C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$.

Theorem 1.25. *a) Let $0 < \gamma < 1$. Assume that $f \in C^1([0,T], L^2(\mathbb{R}, d\mu_\alpha))$ and $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then a generalised solution of the Cauchy problem (1.3) exists, is unique, and given by the expression*

$$
\begin{aligned}
u(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2} t^\gamma\right) g(y) E_\alpha(x,\lambda) E_\alpha(-y,\lambda) \\
\times\, d\mu_\alpha(y) d\mu_\alpha(\lambda) \\
+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t (t-\tau)^{\gamma-1} \mathbb{E}_{\gamma,\gamma}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^\gamma\right) \frac{f(\tau,y)}{1+a\lambda^2} \\
\times\, E_\alpha(x,\lambda) E_\alpha(-y,\lambda) d\tau d\mu_\alpha(y) d\mu_\alpha(\lambda), \quad (1.4)
\end{aligned}
$$

where $\mathbb{E}_{\gamma,1}$ and $\mathbb{E}_{\gamma,\gamma}$ are the Mittag-Leffler functions.
b) Let $\gamma = 1$. Assume that $f \in C([0,T], L^2(\mathbb{R}, d\mu_\alpha))$ and $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then the Cauchy problem (1.3) has a unique generalised solution, which is given by Expression (1.4).

Then we studied inverse source problem for the time-fractional pseudo-parabolic equation is

$$
\begin{cases}
\mathbb{D}^\gamma_{0+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), & (t,x) \in Q_T \\
u(0,x) = \phi(x), \quad x \in \mathbb{R}, \\
u(T,x) = \psi(x), \quad x \in \mathbb{R},
\end{cases}
\tag{1.5}
$$

where ϕ and ψ are given suitable functions, and

$$
\mathbb{D}^\gamma_{0+,t} = \begin{cases} \mathcal{D}^\gamma_{0+,t} & \text{if} \quad 0 < \gamma < 1, \\ \partial_t & \text{if} \quad \gamma = 1. \end{cases}
$$

We assume that $0 < \gamma \leq 1$. Then generalized solution of Inverse source problem (1.5) is the pair of functions (u, f), where $f \in L^2(\mathbb{R}, d\mu_\alpha)$ and

$$
u \in C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)).
$$

Theorem 1.26. *We assume that $\psi, \phi \in \mathcal{H}_\alpha(\mathbb{R}, \mu_\alpha)$. Then a generalized solution of Inverse source problem (1.5) exists, is unique, and can be written by the expressions*

$$
\begin{aligned}
f(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} (m+\lambda^2) \frac{\psi(y) - \phi(y)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} E_\alpha(x,\lambda) E_\alpha(-y,\lambda) \\
\times\, d\mu_\alpha(y) d\mu_\alpha(\lambda)
\end{aligned}
$$

and

$$
u(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} \psi(y) E_\alpha(x,\lambda) E_\alpha(-y,\lambda)
$$

$$\times d\mu_\alpha(y)d\mu_\alpha(\lambda)$$

$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\phi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)$$
$$\times d\mu_\alpha(y)d\mu_\alpha(\lambda).$$

We complete our thesis with inverse source problem for heat equation with the bi-ordinal Hilfer fractional derivative generated by the Dunkl operator (Section 4.3). First, we consider direct problem.

Definition 1.27. We will call the function u a regular solution if it satisfies regularity conditions

$$t^{1-\eta}u(\cdot,x)\in C[0,T],\text{ and }D_{0+,t}^{(\gamma_1,\gamma_2)s}u(\cdot,x),D_{\alpha,x}^2u(\cdot,x)\in C(0,T),$$

and the equation (1.6) for all $(t,x)\in Q_T$, where $\eta:=\gamma_2+\mu(1-\gamma_2)$ and $D_{0+}^{(\gamma_1,\gamma_2)s}$ is the bi-ordinal Hilfer fractional derivative (Definition 2.71).

Let $0<\gamma_1,\gamma_2\leq 1$, $s\in[0,1]$, $a>0$ and $\alpha\geq -1/2$. Our aim is to find a regular solution u of the problem

$$\begin{cases} D_{0+,t}^{(\gamma_1,\gamma_2)s}u(t,x)=aD_{\alpha,x}^2u(t,x)+f(t,x), & (t,x)\in Q_T\\ \lim_{t\to 0+}I_{0+}^{1-\eta}u(t,x)=\xi(x), \quad x\in\mathbb{R}, \end{cases} \tag{1.6}$$

where the functions f and ξ are given functions, and $I_{0+}^{1-\eta}$ is a left-hand sided Riemann-Liouville fractional integrals (Definition 2.66). The following theorem demonstrates the unique solvability of the direct problem.

Theorem 1.28. *We assume that $f\in C([0,T],\mathcal{H})$ and $I_{0+,t}^{\delta}\widehat{f}(t,\lambda)$ is finite for every fixed $\lambda\in\mathbb{R}$, $\xi\in\mathcal{H}$, and $\delta>1/2$. Then Problem (1.6) has a unique regular solution $t^{1-\eta}u\in C([0,T],\mathcal{H})$ and $D_{0+,t}^{(\gamma_1,\gamma_2)s}u\in C([0,T],L^2(\mathbb{R},d\mu_\alpha))$. Moreover it has a expression*

$$u(t,x)=t^{\eta-1}\int_{\mathbb{R}}\widehat{\xi}(\lambda)\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)E_\alpha(x,\lambda)d\mu_\alpha(\lambda)$$
$$+\int_{\mathbb{R}}\left[\int_0^t(t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]\widehat{f}(\tau,\lambda)d\tau\right]E_\alpha(x,\lambda)d\mu_\alpha(\lambda),$$

where $\delta:=\gamma_2+s(\gamma_1-\gamma_2)$.

Then we consider inverse source problem. Let $0<\gamma_1,\gamma_2\leq 1$, $s\in[0,1]$, $a>0$ and $\alpha\geq -1/2$. Our aim is to find a solution pair (u,f) of the inverse source problem

$$\begin{cases} D_{0+,t}^{(\gamma_1,\gamma_2)s}u(t,x)=aD_{\alpha,x}^2u(t,x)+p(t)f(x), \quad x\in\mathbb{R}, & (t,x)\in Q_T\\ \lim_{t\to 0+}I_{0+}^{1-\eta}u(t,x)=\phi(x), \quad x\in\mathbb{R},\\ u(T,x)=\psi(x), \quad x\in\mathbb{R}, \end{cases} \tag{1.7}$$

where the functions p, ϕ and ψ are given functions. For this problem we have the following result.

Theorem 1.29. *Let $\psi, \phi \in \mathcal{H}$. We assume that $p \in C[0,T]$ and*

$$C^* := \int_0^T (T-\tau)^{\delta-1} E_{\delta,\delta}\left[-a\lambda^2 (T-\tau)^\delta\right] p(\tau) d\tau$$

is a finite well defined nonzero number for every $T > 0$ and $\lambda \in \mathbb{R}$, and $\delta > 1/2$. Then Problem (1.7) has a unique solution pair (u, f), where u is a regular solution, which are $f \in L^2(\mathbb{R}, d\mu_\alpha)$ and $t^{1-\eta} u \in C([0,T], \mathcal{H})$ with $D_{0+,t}^{(\gamma_1,\gamma_2)s} u \in C([0,T], L^2(\mathbb{R}, d\mu_\alpha))$, and expressed by

$$\begin{aligned} u(t,x) = t^{\eta-1} \int_{\mathbb{R}} \widehat{\phi}(\lambda) \mathbb{E}_{\delta,\eta}(-a\lambda^2 t^\delta) E_\alpha(x,\lambda) d\mu_\alpha(\lambda) \\ + \int_{\mathbb{R}} \frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda) T^{\eta-1} \mathbb{E}_{\delta,\eta}(-a\lambda^2 T^\delta)}{C^*} \\ \times \left(\int_0^t (t-\tau)^{\delta-1} \mathbb{E}_{\delta,\delta}\left[-a\lambda^2 (t-\tau)^\delta\right] p(\tau) d\tau \right) E_\alpha(x,\lambda) d\mu_\alpha(\lambda) \end{aligned}$$

and

$$f(x) = \frac{1}{C^*} \int_{\mathbb{R}} \left(\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda) T^{\eta-1} \mathbb{E}_{\delta,\eta}(-a\lambda^2 T^\delta) \right) E_\alpha(x,\lambda) d\mu_\alpha(\lambda).$$

2. Preliminary results

In this chapter, we introduce some of the key concepts and techniques of function analysis, fractional calculus, and rational Dunkl theory, which are used in further chapters. These topics include linear operators, function spaces, and convergence in function spaces. We also discuss some of the fundamental theorems in function analysis, such as Lebesgue's dominated convergence theorem and two types of Schwartz kernel theorem. We briefly provide the necessary definitions of fractional differential operators, such as Riemann-Liouville, Caputo, and bi-ordinal Hilfer fractional operators. Additionally, we cover major topics in rational Dunkl theory: the Dunkl operator, the Dunkl kernel, the Dunkl transform, the Dunkl convolution, and generalized Taylor formula. Some of results in Dunkl analysis is proven by ourselves, so they might be new.

2.1. **Background from elementary function analysis.** Let X and Y be vector spaces over the same scalar field $\mathbb{K}$ (here $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$). A mapping A which assigns to each element x of a set $D(A) \subset X$ a unique element $y \in Y$ is called an **operator**. The set $D(A)$ on which A acts is called the domain of A.

Definition 2.1. Let X and Y be vector spaces over the same scalar field $\mathbb{K}$. A function $A \subseteq X \times Y$ is said to be a **linear operator** (or a linear mapping) if

- $D(A)$ is a subspace of X

and

- $A(\alpha x + \beta y) = \alpha A(x) + \beta A(y)$ for every scalars $\alpha, \beta \in \mathbb{K}$ and every vectors $x, y \in X$.

We often write Ax, rather than $A(x)$. A linear mapping $F : X \longrightarrow \mathbb{K}$ is called a **linear functional** and $f : \mathbb{K} \longrightarrow \mathbb{K}$ is called a **function** (real or complex valued).

Definition 2.2. A nonnegative function $x \mapsto p(x)$ on a vector space X is called a **seminorm** if it satisfies the following conditions:

- $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and all $\lambda \in \mathbb{K}$;
- $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Definition 2.3. A seminorm on a vector space X is called a **norm** if

$$\text{for all} \quad x \in X, \quad p(x) = 0 \quad \text{implies} \quad x = 0.$$

Definition 2.4 (**Space** $C_c^\infty(\Omega)$)**.** For an open set $\Omega \subset \mathbb{R}$, the space $C_c^\infty(\Omega)$ of **smooth compactly supported functions** is defined as the space of smooth functions $\varphi : \Omega \longrightarrow \mathbb{K}$ with compact support. Here the support of φ is defined as the closure of the set where φ is non-zero, i.e., by

$$\operatorname{supp} \varphi = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}.$$

Example 2.5. The function

$$\psi(x) = \begin{cases} \exp\left(\frac{1}{x^2 - 1}\right), & \text{for} \quad |x| < 1, \\ 0, & \text{for} \quad |x| \geq 1, \end{cases}$$

with supp $\psi = \{x \in \mathbb{R} : |x| \leq 1\}$ belongs to $\psi \in C_c^\infty(\mathbb{R})$.

Definition 2.6 (**Convergence in $C_c^\infty(\Omega)$**)**.** We say that $\varphi_k \to \varphi$ in $C_c^\infty(\Omega)$ if $\varphi_k, \varphi \in C_c^\infty(\Omega)$, if there is a compact set $K \subset \Omega$ such that supp$\varphi_k \subset K$ for all k, and if

$$\sup_{x\in\Omega} \left| \frac{d^n}{dx^n}(\varphi_k - \varphi)(x) \right| \to 0$$

for all $n \in \mathbb{N}$.

Definition 2.7 (**Distributions $\mathcal{D}'(\Omega)$**)**.** The space $\mathcal{D}'(\Omega)$ is **the space of continuous linear functionals** on $C_c^\infty(\Omega)$. This means that $u \in \mathcal{D}'(\Omega)$ if it is a functional $u : C_c^\infty(\Omega) \longrightarrow \mathbb{K}$ such that:

- u is linear, i.e., $u(\alpha\phi + \beta\psi) = \alpha u(\phi) + \beta u(\psi)$ for all $\alpha, \beta \in \mathbb{K}$ and all $\phi, \psi \in C_c^\infty(\Omega)$;
- u is continuous, i.e., $u(\phi_j) \to u(\phi)$ in $\mathbb{K}$ whenever $\phi_j \to \phi$ in $C_c^\infty(\Omega)$.

Definition 2.8 (**Schwartz space $\mathcal{S}(\mathbb{R})$**)**.** The **Schwartz space** $\mathcal{S}(\mathbb{R})$ is the topological vector space of functions $f : \mathbb{R} \longrightarrow \mathbb{K}$ such that $f \in C^\infty(\mathbb{R})$ and

$$x^k \frac{d^n}{dx^n} f(x) \to 0 \quad \text{as} \quad |x| \to \infty$$

for all $n, k \in \mathbb{N}$. The seminorms on the space $\mathcal{S}(\mathbb{R})$ are defined by

$$p_{n,k}(f) := \sup_{x\in\mathbb{R}} \left| x^k \frac{d^n}{dx^n} f(x) \right| \tag{2.1}$$

for all $n, k \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R})$.

We say that the function belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ if $f \in C^\infty(\mathbb{R})$ and $p_{n,k}(f) < +\infty$. The requirement

$$p_{n,k}(f) = \sup_{x\in\mathbb{R}} \left| x^k \frac{d^n}{dx^n} f(x) \right| < +\infty \tag{2.2}$$

can be replaced by the condition

$$\left| \frac{d^n}{dx^n} f(x) \right| \le \frac{C_{k,n}}{(1+|x|)^k}, \quad \text{for some} \quad C_{k,n} > 0, \tag{2.3}$$

for all $n, k \in \mathbb{N}$ and $x \in \mathbb{R}$. Let inequality (2.3) hold for all $n, k \in \mathbb{N}$ and $x \in \mathbb{R}$. Then we obtain

$$\left| x^k \frac{d^n}{dx^n} f(x) \right| = |x|^k \left| \frac{d^n}{dx^n} f(x) \right| \le (1+|x|)^k \left| \frac{d^n}{dx^n} f(x) \right| \le C_{k,n},$$

which leads (2.2). On the other hand, if (2.2) holds, then Newton's Binomial Theorem gives (2.3).

Example 2.9. The function $f(x) = \exp(-x^2)$ belongs to $\mathcal{S}(\mathbb{R})$. More generally, if p is any polynomial, then $g(x) = p(x)\exp(-x^2)$ belongs to $\mathcal{S}(\mathbb{R})$.

Example 2.10. The function

$$f(x) = \frac{1}{(1+x^2)^k}$$

does not belongs to $\mathcal{S}(\mathbb{R})$ for any $k \in \mathbb{N}$ since $x^{2k} f(x)$ does not decay to zero as $|x| \to \infty$.

Definition 2.11 (**Convergence in $\mathcal{S}(\mathbb{R})$**)**.** We will say that $f_j \to f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$, if $\{f_j\} \subset \mathcal{S}(\mathbb{R})$, $f \in \mathcal{S}(\mathbb{R})$ and if

$$p_{n,k}(f_j - f) \to 0 \quad \text{as} \quad j \to \infty,$$

for all $n, k \in \mathbb{N}$, where seminorms $p_{n,k}$ are defined by the formula (2.1).

Definition 2.12. A linear operator

$$A : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

is continuous if $f_j \to f$ in $\mathcal{S}(\mathbb{R})$ implies $Af_j \to Af$ in $\mathcal{S}(\mathbb{R})$.

Let us give here very useful theorem, formulated as following:

Theorem 2.13 (**Lebesgue's dominated convergence theorem**)**.** [72, Theorem 1.1.4, p. 222] *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}$ such that $f_k \to f$ pointwise almost everywhere on $\mathbb{R}$ as $k \to \infty$. Suppose there is an integrable function $g \in L^1(\mathbb{R})$ such that $|f_k| \leq g$ for all k. Then f is integrable and*

$$\int_{\mathbb{R}} f(x)dx = \lim_{k\to\infty} \int_{\mathbb{R}} f_k(x)dx.$$

Proposition 2.14. [72, Lemma 1.3.32, p. 240] *The space $C_c^{\infty}(\mathbb{R})$ is sequentially dense in $\mathcal{S}(\mathbb{R})$, i.e., for every $\varphi \in \mathcal{S}(\mathbb{R})$ there exists a sequence $\varphi_k \in C_c^{\infty}(\mathbb{R})$ such that $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R})$ as $k \to \infty$.*

Definition 2.15 (**Tempered distributions $\mathcal{S}'(\mathbb{R})$**)**.** We define the **space of tempered distributions** $\mathcal{S}'(\mathbb{R})$ as the space of all continuous linear functionals on $\mathcal{S}(\mathbb{R})$. This means that $u \in \mathcal{S}'(\mathbb{R})$ if it is a functional $u : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{K}$ such that:

- u is linear, i.e., $u(\alpha\phi + \beta\psi) = \alpha u(\phi) + \beta u(\psi)$ for all $\alpha, \beta \in \mathbb{K}$ and all $\phi, \psi \in \mathcal{S}(\mathbb{R})$;
- u is continuous, i.e., $u(\phi_j) \to u(\phi)$ in $\mathbb{K}$ whenever $\phi_j \to \phi$ in $\mathcal{S}(\mathbb{R})$.

We can also define the convergence in the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions. Let $u_j, u \in \mathcal{S}'(\mathbb{R})$. We will say that $u_j \to u$ in $\mathcal{S}'(\mathbb{R})$ as $j \to \infty$ if $u_j(\phi) \to u(\phi)$ in $\mathbb{K}$ as $j \to \infty$, for all $\phi \in \mathcal{S}(\mathbb{R})$. Functions in $\mathcal{S}(\mathbb{R})$ are called the test functions for tempered distributions in $\mathcal{S}'(\mathbb{R})$. Another notation for $u(\varphi)$ will be $\langle u, \varphi \rangle$.

Example 2.16. The function $I : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{R}$ defined by the Riemann integral

$$I(\phi) = \int_0^1 \phi(x)dx$$

is a tempered distribution.

Proposition 2.17 (**Continuous inclusion $\mathcal{S}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$**)**.** [72, Exercise 1.4.6, p. 242] *The inclusion $\mathcal{S}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ is continuous, i.e. $u_k \to u$ in $\mathcal{S}'(\mathbb{R})$ implies $u_k \to u$ in $\mathcal{D}'(\mathbb{R})$.*

The following Theorems proved in H. Gask's work [36] published in 1960.

Theorem 2.18 (**Schwartz kernel theorem I**)**.** *For any separately continuous bilinear functional A on $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ there exists precisely one distribution K in $\mathcal{S}'(\mathbb{R} \times \mathbb{R})$ such that*

$$A(f, g) = \langle K, fg \rangle$$

for all (f, g) in $\mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$.

Theorem 2.19 (**Schwartz kernel theorem II**). *For any separately continuous bilinear functional A on $C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R})$ there exists precisely one distribution K in $\mathcal{D}'(\mathbb{R} \times \mathbb{R})$ such that*

$$A(f, g) = \langle K, fg \rangle$$

for all (f, g) in $C_c^\infty(\mathbb{R}) \times C_c^\infty(\mathbb{R})$.

Now, let us provide some necessary information about **Gamma function** and **Bessel functions**.

The **Gamma function** is a special function, which is defined for all complex numbers except the non-positive integers. For complex numbers z such that $Re(z) > 0$ the gamma function Γ is given by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx.$$

We have for the Gamma function the following recurrence relation:

$$\Gamma(z+1) = z\Gamma(z). \tag{2.4}$$

We can show (2.4) by Integration by parts:

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty e^{-x} x^{z-1} dx \\
&= \int_0^\infty e^{-x} \left(\frac{x^z}{z}\right)' dx \\
&= e^{-x} \cdot \frac{x^z}{z}\Big|_0^\infty + \int_0^\infty e^{-x} \frac{x^z}{z} dx \\
&= \frac{1}{z} \int_0^\infty e^{-x} x^z dx \\
&= \frac{1}{z} \cdot \Gamma(z+1).
\end{aligned}$$

From (2.4) we obtain an interesting formula

$$\frac{\Gamma(n+z+1)}{\Gamma(z+1)} = (n+z)(n-1+z) \cdot \ldots \cdot (1+z). \tag{2.5}$$

The Gamma function is the generalisation of the factorial to complex numbers. Indeed,

$$\Gamma(n+1) = n!, \quad n \in \mathbb{N}.$$

Example 2.20. $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.

Example 2.21. $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$ for all $n \in \mathbb{Z}^+$.

The **Bessel functions** are canonical solutions y of Bessel's differential equation

$$z^2 \frac{d^2 y}{dz^2} + z\frac{dy}{dz} + (z^2 - \alpha^2) y = 0$$

for an arbitrary $\alpha \in \mathbb{C}$. Solutions of the Bessel's differential equation can be found by applying the Frobenius method, which have the series expression given by

$$J_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n+\alpha} \tag{2.6}$$

and are called the Bessel functions of the first kind.

The following theorem will be useful for us:

Theorem 2.22. ([99, Theorem 7.5, p. 39]) *Between two consecutive positive zeros of the Bessel function $J_\alpha(x\lambda)$, $\alpha > 0$ there is one and only one root of the Bessel function $J_{\alpha+1}(x\lambda)$, and vice versa.*

There are another type of Bessel functions of the first kind, which are called the normalized Bessel function of the first kind and are denoted by $j_\alpha(z)$. The functions $j_\alpha(z)$ have the following series representation

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{z}{2}\right)^{2n}. \tag{2.7}$$

If $z = x\lambda$, where $x \in \mathbb{R}, \lambda \in \mathbb{C}$, then the functions $j_\alpha(x\lambda)$ has the following properties ([93, p. 9]):

- For all $\lambda \in \mathbb{C}$, the function $x \longmapsto j_\alpha(x\lambda)$ is an even C^∞-function on $\mathbb{R}$;
- For all $x \in \mathbb{R}$, the function $\lambda \longmapsto j_\alpha(x\lambda)$ is an even entire function on $\mathbb{C}$.

2.2. **The Dunkl analysis.** In this subsection, we provide basic definitions and facts from Dunkl analysis. Some results, such as Lemma 2.39, are proven by ourselves, as we are uncertain whether such a lemma already exists (though it seems likely, as it is basic).

Definition 2.23. ([74, Examples 2.2, p.99]) The **Dunkl operator** is the differential-difference operator

$$D_\alpha : C^1(\mathbb{R}) \longrightarrow C(\mathbb{R})$$

defined by

$$D_\alpha f(x) = \frac{d}{dx} f(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}, \tag{2.8}$$

for every $\alpha \geq -\frac{1}{2}$ and the **Dunkl Laplacian**

$$D_\alpha^2 : C^2(\mathbb{R}) \longrightarrow C(\mathbb{R})$$

is defined by

$$D_\alpha^2 f(x) = \frac{d^2}{dx^2} f(x) + \frac{2\alpha+1}{x} \frac{d}{dx} f(x) - \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x^2}$$

for every $\alpha \geq -\frac{1}{2}$. Here we understand D_α^2 as a composition of the operators D_α and D_α, i.e. $D_\alpha^2 = D_\alpha D_\alpha$.

Remark 2.24. From the Mean Value Theorem we readily see that

$$\frac{f(x) - f(-x)}{x} = 2\frac{f(x) - f(-x)}{x - (-x)} = 2f'(c)$$

for some $-x < c < x$. This gives

$$D_\alpha f(x) = \frac{d}{dx}f(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} = f'(x) + (2\alpha+1)f'(c) \tag{2.9}$$

for some $-x < c < x$ and

$$\sup_{x\in\mathbb{R}} |D_\alpha f(x)| \leq 2(\alpha+1) \sup_{x\in\mathbb{R}} |f'(x)|.$$

Also the expression (2.9) gives a following result

$$\sup_{x\in\mathbb{R}} \left|x^k D_\alpha^n f(x)\right| \leq 2(\alpha+1) \sup_{x\in\mathbb{R}} \left|x^k \frac{d^n}{dx^n} f(x)\right| < +\infty \tag{2.10}$$

for $f \in \mathcal{S}(\mathbb{R})$.

Remark 2.25. In general, the Dunkl operator is defined for every $\alpha \in \mathbb{C}$, but in this thesis we are interested only real $\alpha \geq -\frac{1}{2}$.

The Dunkl operator D_α is not only well defined from $C^1(\mathbb{R})$ to $C(\mathbb{R})$, we can consider as a domain of D_α more important spaces, as $C^m(\mathbb{R})$ with $m \geq 1$, $C^\infty(\mathbb{R})$, $C_c^\infty(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$.

Lemma 2.26. ([74, Lemma 2.2, p.6]) *If $f \in C^m(\mathbb{R})$ with $m \geq 1$, then $D_\alpha f \in C^{m-1}(\mathbb{R})$.*

Lemma 2.27. ([6, Proposition 3.4, p.28]) *The Dunkl operators map the following function spaces into themselves:*

$$C^\infty(\mathbb{R}), C_c^\infty(\mathbb{R}) \quad \text{and} \quad \mathcal{S}(\mathbb{R}).$$

Proposition 2.28. [17, Proposition 2.1, p. 103] *Let $\alpha > -\frac{1}{2}$. For $\lambda \in \mathbb{C}$, the following differential equation with initial condition:*

$$D_\alpha f(x) = i\lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R} \tag{2.11}$$

has a unique solution $E_\alpha(x,\lambda)$ given by

$$E_\alpha(x,\lambda) = j_\alpha(x\lambda) + i\frac{x\lambda}{2(\alpha+1)} j_{\alpha+1}(x\lambda), \tag{2.12}$$

where j_α is called the normalized Bessel function of first kind.

Bewijs. Let f be a solution of the problem (2.11). We can write

$$f(x) = u(x) + v(x), \quad x \in \mathbb{R},$$

where

$$u(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad v(x) = \frac{f(x) - f(-x)}{2}.$$

Then after putting f to the equation (2.11) we obtain an equivalent equation

$$\frac{d}{dx}v(x) + \frac{2\alpha+1}{x}v(x) = i\lambda u(x), \quad \frac{d}{dx}u(x) = i\lambda v(x), \quad u(0) = 1.$$

So, u satisfies Bessel type equation

$$\frac{d^2}{dx^2}u(x) + \frac{2\alpha+1}{x}\frac{d}{dx}u(x) = -\lambda^2 u(x), \quad u(0) = 1, \quad \frac{d}{dx}u(0) = 0. \tag{2.13}$$

The Bessel type equation (2.13) has the unique solution ([93, p. 10])

$$x \longmapsto j_\alpha(x\lambda)$$

for all $\lambda \in \mathbb{C}$, which is called the normalized Bessel function of first kind (2.7). Derivative of the function $x \longmapsto j_\alpha(x\lambda)$ gives us

$$\begin{aligned}\frac{d}{dx}u(x) &= \frac{d}{dx}j_\alpha(x\lambda)\\ &= \Gamma(\alpha+1)\sum_{n=1}^{\infty}\frac{(-1)^n\lambda}{(n-1)!\Gamma(n+\alpha+1)}\left(\frac{x\lambda}{2}\right)^{2n-1}\\ &= -\frac{x\lambda^2}{2(\alpha+1)}\Gamma(\alpha+2)\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+2)}\left(\frac{x\lambda}{2}\right)^{2m}\\ &= -\frac{x\lambda^2}{2(\alpha+1)}j_{\alpha+1}(x\lambda).\end{aligned}$$

Thus, we have

$$v(x) = -\frac{x\lambda}{2i(\alpha+1)}j_{\alpha+1}(x\lambda) = \frac{ix\lambda}{2(\alpha+1)}j_{\alpha+1}(x\lambda),$$

from which we obtain (2.12). □

So, the function $E_\alpha(x,\lambda)$ has the following properties:

- For all $\lambda \in \mathbb{C}$, the function $x \longmapsto E_\alpha(x,\lambda)$ is a C^∞-function on $\mathbb{R}$;
- For all $x \in \mathbb{R}$, the function $\lambda \longmapsto E_\alpha(x,\lambda)$ is an entire function on $\mathbb{C}$;
- $E_\alpha(x,\lambda) = E_\alpha(\lambda,x)$;
- $E_\alpha(\xi x,\lambda) = E_\alpha(x,\xi\lambda)$,

where $\xi \in \mathbb{C}$.

Remark 2.29. In the case $\alpha = -\frac{1}{2}$, the equation (2.11) turns into an ODE

$$D_{-\frac{1}{2}}f(x) = \frac{d}{dx}f(x) = i\lambda f(x), \quad f(0)=1, \quad x\in\mathbb{R}, \quad \lambda\in\mathbb{C},$$

which has a solution

$$f(x) = \exp(ix\lambda).$$

Remark 2.30. The function $E_\alpha(x,\lambda)$ is called **the Dunkl kernel** in the literature.

Corollary 2.31. [68, Formulas 2.2.4-2.2.6, p. 371] *Let $\alpha \geq -\frac{1}{2}$. Then the Dunkl kernel has the following series representation*

$$E_\alpha(x,\lambda) = \sum_{n=0}^{\infty}\frac{(ix\lambda)^n}{\gamma_\alpha(n)}, \tag{2.14}$$

where γ_α is a generalized factorial, defined by

$$\gamma_\alpha(2n) = \frac{2^{2n}n!\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \quad \text{and} \quad \gamma_\alpha(2n+1) = \frac{2^{2n+1}n!\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)}. \tag{2.15}$$

Bewijs. The proposition follows from next short calculations

$$
\begin{aligned}
E_\alpha(x,\lambda) &= j_\alpha(x\lambda) + i\frac{x\lambda}{2(\alpha+1)} j_{\alpha+1}(x\lambda) \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{x\lambda}{2}\right)^{2n} + i\frac{x\lambda}{2(\alpha+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+2)(-1)^n}{n!\Gamma(n+\alpha+2)} \left(\frac{x\lambda}{2}\right)^{2n} \\
&= \sum_{n=0}^{\infty} \left(\frac{\Gamma(\alpha+1)}{2^{2n} n!\Gamma(n+\alpha+1)} (ix\lambda)^{2n} + \frac{\Gamma(\alpha+1)}{2^{2n+1} n!\Gamma(n+\alpha+2)} (ix\lambda)^{2n+1} \right).
\end{aligned}
$$

□

Remark 2.32. When $\alpha = -\frac{1}{2}$, we obtain $\gamma_{-\frac{1}{2}}(2n) = (2n)!$ and $\gamma_{-\frac{1}{2}}(2n+1) = (2n+1)!$, indeed

$$
\gamma_{-\frac{1}{2}}(2n) = \frac{2^{2n} n! \Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{2^{2n} n!(2n)!\sqrt{\pi}}{4^n n!\sqrt{\pi}} = (2n)!
$$

and

$$
\begin{aligned}
\gamma_{-\frac{1}{2}}(2n+1) = \frac{2^{2n+1} n! \Gamma\left(n+\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)} &= \frac{2^{2n+1} n! \left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \\
&= \frac{2^{2n} n!(2n)!(2n+1)\sqrt{\pi}}{4^n n!\sqrt{\pi}} = (2n+1)!,
\end{aligned}
$$

where we have used properties of Gamma function.

Remark 2.33. [68, Formulas 2.2.7, p. 372] The generalized factorial γ_α also has a recurrent formula

$$
\gamma_\alpha(n+1) = (n+1+(2\alpha+1)\theta_{n+1})\gamma_\alpha(n), \tag{2.16}
$$

where θ_{n+1} is 0 for even $n+1$ and 1 for odd $n+1$.

Lemma 2.34. *Suppose that $\alpha \geq -\frac{1}{2}$ and $n \in \mathbb{N}$. Then*

$$
n! \leq \gamma_\alpha(n). \tag{2.17}
$$

Bewijs. The Lemma can be proved by using mathematical induction. The Basis for induction is clearly true, since

$$
1 = 0! = \gamma_\alpha(0) = 1 \quad \text{and} \quad \gamma_\alpha(1) = 2(\alpha+1) \geq 2(-\frac{1}{2}+1) = 1 = 1!
$$

For induction step, suppose $k! \leq \gamma_\alpha(k)$ is true. Then inequality (2.17) holds for $k+1$, which is clear from

$$
\gamma_\alpha(k+1) = (k+1+(2\alpha+1)\theta_{k+1})\gamma_\alpha(k) \geq (k+1)! + (2\alpha+1)\theta_{k+1}k! \geq (k+1)!
$$

So, the induction step holds. □

Corollary 2.35. *Let $\alpha = -\frac{1}{2}$. Then Corollary 2.31 leads that*

$$
E_{-\frac{1}{2}}(x,\lambda) = \exp(ix\lambda).
$$

Bewijs. A short calculation. Here we have used property of the Gamma function and (2.14). □

Lemma 2.36. *Assume that $\alpha \geq -\frac{1}{2}$ and $k \in \mathbb{N}$. Then*

$$\left|\frac{d^k}{dx^k}E_\alpha(x,\lambda)\right| \leq |\lambda|^k \exp(|x\lambda|)$$

for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. In particular $|E_\alpha(x,\lambda)| \leq \exp(|x\lambda|)$, when $k = 0$.

Bewijs. Using expression (2.14) of the Dunkl kernel and Lemma 2.34 we obtain

$$\begin{aligned}|E_\alpha(x,\lambda)| = \left|\sum_{k=0}^{\infty}\frac{(ix\lambda)^k}{\gamma_\alpha(k)}\right| = \left|\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(ix\lambda)^k}{\gamma_\alpha(k)}\right| = \lim_{n\to\infty}\left|\sum_{k=0}^{n}\frac{(ix\lambda)^k}{\gamma_\alpha(k)}\right| \\ \leq \lim_{n\to\infty}\sum_{k=0}^{n}\frac{|x\lambda|^k}{\gamma_\alpha(k)} \leq \lim_{n\to\infty}\sum_{k=0}^{n}\frac{|x\lambda|^k}{n!} = \exp(|z\lambda|).\end{aligned}$$

Differentiation k times from $x \longmapsto E_\alpha(x,\lambda)$ gives us

$$\frac{d^k}{dx^k}E_\alpha(x,\lambda) = \frac{d^k}{dx^k}\left(\sum_{n=0}^{\infty}\frac{(ix\lambda)^n}{\gamma_\alpha(n)}\right) = \sum_{n=k}^{\infty}\frac{n(n-1)...(n-k+1)(ix\lambda)^{n-k}\lambda^k}{\gamma_\alpha(n)}.$$

Taking into account this fact we reach the inequality

$$\begin{aligned}\left|\frac{d^k}{dx^k}E_\alpha(x,\lambda)\right| &\leq \sum_{n=k}^{\infty}\frac{n(n-1)...(n-k+1)|x\lambda|^{n-k}|\lambda|^k}{\gamma_\alpha(n)} \\ &\leq |\lambda|^k\sum_{n=k}^{\infty}\frac{n(n-1)...(n-k+1)|x\lambda|^{n-k}}{n!} \\ &= |\lambda|^k\sum_{n=k}^{\infty}\frac{|x\lambda|^{n-k}}{(n-k)!} = |\lambda|^k\sum_{m=0}^{\infty}\frac{|x\lambda|^m}{m!}.\end{aligned}$$

□

Proposition 2.37. *Let $\alpha > -\frac{1}{2}$. Then for every $\lambda \in \mathbb{C}$, the Dunkl kernel has the Poisson integral representation*

$$E_\alpha(x,\lambda) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t)^{\alpha-\frac{1}{2}}(1+t)^{\alpha+\frac{1}{2}}\exp(ix\lambda t)dt \tag{2.18}$$

for all $x \in \mathbb{R}$.

Bewijs. Let $\alpha > -\frac{1}{2}$. Then for every $\lambda \in \mathbb{C}$, the function $j_\alpha(x\lambda)$ has the Poisson integral representation

$$j_\alpha(x\lambda) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\exp(ix\lambda t)dt$$

for all $x \in \mathbb{R}$ ([93, Formula 1.II.12, p. 11]). Thus,

$$\begin{aligned}E_\alpha(x,\lambda) = j_\alpha(x\lambda) + \frac{1}{i\lambda}\frac{d}{dx}j_\alpha(x\lambda) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\exp(ix\lambda t)dt \\ + \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}t\exp(ix\lambda t)dt\end{aligned}$$

$$= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t)^{\alpha-\frac{1}{2}}(1+t)^{\alpha+\frac{1}{2}} \exp(ix\lambda t)dt.$$

□

From integral representation of the Dunkl kernel (2.18) it is convenient to obtain another property of the Dunkl kernel

$$\overline{E_\alpha(x,\lambda)} = E_\alpha(-x,\overline{\lambda}) = E_\alpha(x,-\overline{\lambda}). \tag{2.19}$$

Corollary 2.38. *Let $\alpha > -\frac{1}{2}$ and $\lambda \in \mathbb{R}$. Then we have the following estimates for the Dunkl kernel*

$$\left|\frac{d^k}{dx^k}E_\alpha(x,\lambda)\right| \le |\lambda|^k \quad \text{and} \quad \left|\frac{d^k}{d\lambda^k}E_\alpha(x,\lambda)\right| \le |x|^k$$

for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. In particular we have

$$|E_\alpha(x,\lambda)| \le 1 \tag{2.20}$$

for all $x, \lambda \in \mathbb{R}$, when $k = 0$.

Bewijs. A short calculation. Here we have used integral representation of the Dunkl kernel (2.18) and the fact

$$1 = E_\alpha(0,\lambda) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t)^{\alpha-\frac{1}{2}}(1+t)^{\alpha+\frac{1}{2}} dt.$$

□

Now, we are able to give a short representation for Dunkl kernel

$$E_\alpha(x,\lambda) = V_\alpha \exp(ixy)$$

where V_α is the **Dunkl's intertwining operator**, defined by

$$V_\alpha f(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t)^{\alpha-\frac{1}{2}}(1+t)^{\alpha+\frac{1}{2}} f(xt)dt$$

on the space of smooth functions, i.e. $f \in C^\infty(\mathbb{R})$.

Lemma 2.39. *Let $\alpha \ge -\frac{1}{2}$. The function $E_\alpha(x,\lambda)$ does not have zeros for all $x, y \in \mathbb{R}$.*

Bewijs. (1) Let first $\alpha = -\frac{1}{2}$, then the Dunkl kernel is exponential function

$$E_{-\frac{1}{2}}(x,\lambda) = e^{ix\lambda}.$$

So we have to consider two cases $\alpha > 0$ and $-\frac{1}{2} < \alpha < 0$. Using definition of the Bessel function of the first kind (2.6), we can rewrite the Dunkl kernel (2.12) as

$$E_\alpha(x,\lambda) = \frac{2^\alpha\Gamma(\alpha+1)}{(x\lambda)^\alpha}\left[J_\alpha(x\lambda) + iJ_{\alpha+1}(x\lambda)\right].$$

Here we need not worry about $x\lambda = 0$, because $E_\alpha(0) = 1$ and we are going to consider only positive $x\lambda$, it is obvious form equation $J_\alpha(-x) = (-1)^\alpha J_\alpha(x)$.
(2) Now, let us consider case when $\alpha > 0$. Theorem 2.22 implies that the Bessel functions $J_\alpha(x\lambda)$ and $J_{\alpha+1}(x\lambda)$ have many zeros, but they can not be equal to zero at the same time. Thus, $E_\alpha(x,\lambda)$ in not equal to zero for all $x, \lambda > 0$, when $\alpha > 0$.

(3) Finally, in a case $-\frac{1}{2} < \alpha < 0$, we prove lemma using recurrent formula ([99, Formula (3.10), p. 20])

$$J_{\alpha-1}(x\lambda) + J_{\alpha+1}(x\lambda) = \frac{2\alpha}{x\lambda} J_\alpha(x\lambda)$$

as following. Let us fix arbitrary $-\frac{1}{2} < \alpha < 0$, and for some $x_0 > 0$ we have

$$\begin{aligned} &J_{\alpha-1}(x_0) = J_\alpha(x_0) = 0 \quad \Longrightarrow \quad J_{\alpha+1}(x_0) = 0, \\ &J_{\alpha+1}(x_0) = J_{\alpha+2}(x_0) = 0 \quad \Longrightarrow \quad J_{\alpha+3}(x_0) = 0, \\ &\dots \end{aligned}$$

If we continue this process until $\alpha > 0$, we obtain $J_\alpha(x_0) = J_{\alpha+1}(x_0) = 0$ for some $x_0 > 0$ and it is contradiction to Theorem 2.22. □

2.2.1. *The Dunkl transform.* Let $L^p(\mathbb{R}, d\mu_\alpha), 1 \le p \le +\infty$, be the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}} < +\infty, \quad \text{if} \quad 1 \le p < +\infty$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < +\infty, \quad \text{if} \quad p = +\infty,$$

where

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx. \tag{2.21}$$

Remark 2.40. Note that, in the case $\alpha = -1/2$, expression (2.21) gives us

$$d\mu_{-\frac{1}{2}}(x) = \frac{1}{\sqrt{2\pi}} dx,$$

thus $L^p(\mathbb{R}, d\mu_{-\frac{1}{2}})$ is the usual $L^p(\mathbb{R})$ space.

Lemma 2.41 (Hölder's inequality). *We assume that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f \in L^p(\mathbb{R}, d\mu_\alpha)$, $g \in L^q(\mathbb{R}, d\mu_\alpha)$. *Then we have*

$$\int_{\mathbb{R}} |f(x)g(x)| d\mu_\alpha(x) \le \|f\|_{p,\alpha} \|g\|_{q,\alpha}. \tag{2.22}$$

Bewijs. Let $f \in L^p(\mathbb{R}, d\mu_\alpha)$ and $g \in L^q(\mathbb{R}, d\mu_\alpha)$. Then

$$\begin{aligned} \int_{\mathbb{R}} |f(x)g(x)| d\mu_\alpha(x) &= \int_{\mathbb{R}} |f(x)g(x)| \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx \\ &= \int_{\mathbb{R}} |f(x)g(x)| \left(\frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} \right)^{\frac{1}{p}} \left(\frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} \right)^{\frac{1}{q}} dx \\ &\le \left(\int_{\mathbb{R}} |f(x)|^p \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(x)|^q \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx \right)^{\frac{1}{q}}, \end{aligned}$$

where we have used classical Hölder's inequality. □

Lemma 2.42. *We have* $\mathcal{S}(\mathbb{R}) \subset L^p(\mathbb{R}, d\mu_\alpha)$ *with continuous embedding, i.e.,* $f_j \to f$ *in* $\mathcal{S}(\mathbb{R})$ *implies that* $f_j \to f$ *in* $L^p(\mathbb{R}, d\mu_\alpha)$ *for all* $1 \le p \le \infty$.

Bewijs. Let $f \in \mathcal{S}(\mathbb{R})$. Then we obtain

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) &\leq \int_{\mathbb{R}} \frac{1}{(1+|x|)^{pk}}(1+|x|)^{pk}|f(x)|^p d\mu_\alpha(x)\\ &\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}} \frac{|x|^{2\alpha+1}}{(1+|x|)^{pk}}dx \cdot \sup_{x\in\mathbb{R}}\big((1+|x|)^k|f(x)|\big)^p < +\infty,\end{aligned}$$

where $pk > 2(\alpha+1)$ and $k \in \mathbb{N}$. Let $f_j \to f$ in $\mathcal{S}(\mathbb{R})$. We assume that $1 \leq p < \infty$. Then we are able to calculate

$$\begin{aligned}\|f_j - f\|_{p,\alpha} &= \int_{\mathbb{R}} |f_j(x) - f(x)|^p \, d\mu_\alpha(x)\\ &\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}} \frac{|x|^{2\alpha+1}}{(1+|x|)^{pk}}dx \cdot \sup_{x\in\mathbb{R}}\big((1+|x|)^k|f_j(x)-f(x)|\big)^p\\ &\to 0.\end{aligned}$$

Now, let $p = \infty$. Then

$$\|f_j - f\|_\infty = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f_j(x) - f(x)| = p_{0,0}(f_i - f) \to 0$$

since $f_j \to f$ in $\mathcal{S}(\mathbb{R})$. □

The Dunkl kernel leads to the **Dunkl transform** $\mathcal{F}_\alpha$, which is defined by the formula

$$\mathcal{F}_\alpha[f](\lambda) = \int_{\mathbb{R}} E_\alpha(-x,\lambda) f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R} \tag{2.23}$$

for $f \in L^1(\mathbb{R}, d\mu_\alpha)$ and using (2.20) we obtain

$$|\mathcal{F}_\alpha[f](\lambda)| \leq \int_{\mathbb{R}} |E_\alpha(-x,\lambda)f(x)| \, d\mu_\alpha(x) \leq \int_{\mathbb{R}} |f(x)| \, d\mu_\alpha(x) = \|f\|_{1,\alpha}$$

for any $\lambda \in \mathbb{R}$. Thus

$$\|\mathcal{F}_\alpha[f]\|_\infty \leq \|f\|_{1,\alpha}.$$

Remark 2.43. For $\alpha = -1/2$, the Dunkl transform $\mathcal{F}_{-\frac{1}{2}}$ is the Fourier transform

$$\mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x) e^{-ix\lambda} dx, \quad \lambda \in \mathbb{R}.$$

The **inverse Dunkl transform** is defined by

$$\mathcal{F}_\alpha^{-1}[f](\lambda) = \mathcal{F}_\alpha[f](-\lambda) = \int_{\mathbb{R}} E_\alpha(x,\lambda) f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}. \tag{2.24}$$

Theorem 2.44. *(1)* [26, Corollary 4.22, p. 159] *The Dunkl transform is a homeomorphism of* $\mathcal{S}(\mathbb{R})$*;*
(2) (Plancherel theorem) [26, Theorem 4.26, p. 160] *The Dunkl transform has a unique extension to an isometric isomorphism of* $L^2(\mathbb{R}, d\mu_\alpha)$*, i.e.*

$$\|\mathcal{F}_\alpha[f]\|_{2,\alpha} = \|f\|_{2,\alpha}$$

for all $f \in L^2(\mathbb{R}, d\mu_\alpha)$*;*
(3) (Inverse Dunkl transform) [26, Theorem 4.20, p. 159] *For all* $f \in L^1(\mathbb{R}, d\mu_\alpha)$ *with* $\mathcal{F}_\alpha[f] \in L^1(\mathbb{R}, d\mu_\alpha)$*,*

$$f(x) = \mathcal{F}_\alpha^{-1}[\mathcal{F}_\alpha[f]](x) \quad a.e.$$

The space $C_0(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ which vanish at infinity.

Lemma 2.45 (Riemann-Lebesgue lemma). [26, Corollary 4.7, p. 156] *The Dunkl transform maps* $L^1(\mathbb{R}, d\mu_\alpha)$ *into* $C_0(\mathbb{R})$.

Proposition 2.46 (Product Rule for the Dunkl operator). *If* $f \in C^1(\mathbb{R})$ *is even, then* $D_\alpha f(x) = f'(x)$. *If* $f, g \in C^1(\mathbb{R})$ *and at least one of them even, then*

$$D_\alpha(f(x) \cdot g(x)) = D_\alpha f(x) \cdot g(x) + f(x) \cdot D_\alpha g(x). \tag{2.25}$$

Bewijs. The first statement of the proposition is obvious, because it follows from the definition of the Dunkl operator (2.8). Without losing generality, we may assume that f is even function, then the second statement of the proposition follows from the following equations

$$\begin{aligned} D_\alpha(f(x) \cdot g(x)) &= \frac{d}{dx}(f(x) \cdot g(x)) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) \cdot g(x) - f(-x) \cdot g(-x)}{x} \\ &= f'(x) \cdot g(x) + f(x) \cdot g'(x) + \left(\alpha + \frac{1}{2}\right) \frac{g(x) - g(-x)}{x} f(x) \\ &= f'(x) \cdot g(x) + f(x) \cdot D_\alpha g(x) \end{aligned}$$

□

Natural question, after Proposition 2.46, is what if one of the functions is odd? Answer to this question is

$$D_\alpha(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot D_\alpha g(x) + (2\alpha + 1) \frac{f(x) g(-x)}{x}$$

if f is odd function. Now, we are ready to define Product Rule for the Dunkl operator for any relevant function. As we can write any function in a form

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f_{even}(x) + f_{odd}(x)$$

we obtain

$$\begin{aligned} D_\alpha(f(x)g(x)) &= D_\alpha(f_{even}(x)g(x) + f_{odd}(x)g(x)) \\ &= D_\alpha(f_{even}(x)g(x)) + D_\alpha(f_{odd}(x)g(x)) \\ &= f'_{even}(x) \cdot g(x) + f_{even}(x) \cdot D_\alpha g(x) + f'_{odd}(x) \cdot g(x) + f_{odd}(x) \cdot D_\alpha g(x) \\ &+ (2\alpha + 1) \frac{f_{odd}(x) g(-x)}{x} \\ &= f'(x) \cdot g(x) + f(x) \cdot D_\alpha g(x) + (2\alpha + 1) \frac{f_{odd}(x) g(-x)}{x} \\ &= f'(x) \cdot g(x) + f(x) \cdot D_\alpha g(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x)g(-x) - f(-x)g(-x)}{x}. \end{aligned}$$

Thus, Product Rule for the Dunkl operator is

$$D_\alpha(f(x)g(x)) = f'(x) \cdot g(x) + f(x) \cdot D_\alpha g(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x)g(-x) - f(-x)g(-x)}{x}$$

for all $f, g \in C^1(\mathbb{R})$.

Proposition 2.47. *Let $\alpha \geq -\frac{1}{2}$. Then for every $f \in \mathcal{S}(\mathbb{R})$ and $g \in C_b^1(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} D_\alpha f(x) \cdot g(x) w_\alpha(x) dx = -\int_{\mathbb{R}} f(x) \cdot D_\alpha g(x) w_\alpha(x) dx \tag{2.26}$$

or

$$\int_{\mathbb{R}} (D_\alpha f(x) \cdot g(x) + f(x) \cdot D_\alpha g(x)) w_\alpha(x) dx = 0,$$

where $w_\alpha(x) = |x|^{2\alpha+1}/(2^{\alpha+1}\Gamma(\alpha+1))$ and $w_\alpha(x)dx = d\mu_\alpha(x)$.

Bewijs. Using definition of the Dunkl operator we obtain

$$D_\alpha f(x) \cdot g(x) + f(x) \cdot D_\alpha g(x) = \frac{d}{dx} f(x) \cdot g(x) + f(x) \cdot \frac{d}{dx} g(x)$$
$$+ (2\alpha+1)\frac{f(x)g(x)}{x} - \left(\alpha + \frac{1}{2}\right)\left[\frac{f(-x)g(x) + f(x)g(-x)}{x}\right].$$

Then we calculate

$$\int_{\mathbb{R}} \frac{d}{dx}(f(x)g(x)) w_\alpha(x) dx = \int_{-\infty}^{0} \frac{d}{dx}(f(x)g(x)) \frac{(-x)^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx$$
$$+ \int_0^\infty \frac{d}{dx}(f(x)g(x)) \frac{x^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dx$$

$$= \int_{-\infty}^{0} f(x)g(x) \frac{2\alpha+1}{2^{\alpha+1}\Gamma(\alpha+1)} (-x)^{2\alpha} dx - \int_0^\infty f(x)g(x) \frac{2\alpha+1}{2^{\alpha+1}\Gamma(\alpha+1)} x^{2\alpha} dx$$
$$= -\int_{-\infty}^{0} \frac{f(x)g(x)}{x} \frac{2\alpha+1}{2^{\alpha+1}\Gamma(\alpha+1)} (-x)^{2\alpha+1} dx - \int_0^\infty \frac{f(x)g(x)}{x} \frac{2\alpha+1}{2^{\alpha+1}\Gamma(\alpha+1)} x^{2\alpha+1} dx$$
$$= -(2\alpha+1) \int_{\mathbb{R}} \frac{f(x)g(x)}{x} w_\alpha(x) dx$$

while the other parts yields

$$(2\alpha+1) \int_{\mathbb{R}} \frac{f(x)g(x)}{x} w_\alpha(x) dx - \left(\alpha + \frac{1}{2}\right) \int_{\mathbb{R}} \frac{f(-x)g(x) + f(x)g(-x)}{x} w_\alpha(x) dx.$$

We can finish our proof taking into account that our last integral equals to zero, since under integral we have an odd function. □

Corollary 2.48. *Let $\alpha \geq -\frac{1}{2}$. Then for every $f \in \mathcal{S}(\mathbb{R})$ and $g \in C^\infty(\mathbb{R})$ we obtain*

$$\int_{\mathbb{R}} D_\alpha^n f(x) \cdot g(x) w_\alpha(x) dx = (-1)^n \int_{\mathbb{R}} f(x) \cdot D_\alpha^n g(x) w_\alpha(x) dx. \tag{2.27}$$

for any $n \in \mathbb{N}$.

Let $f \in \mathcal{S}(\mathbb{R})$. Then the Dunkl transform has the following properties ([74, Lemma 2.6, p. 109]):

- $\mathcal{F}_\alpha[f] \in C^\infty(\mathbb{R})$ and $D_\alpha \mathcal{F}_\alpha[f] = -\mathcal{F}_\alpha[ixf]$;
- $\mathcal{F}_\alpha[D_\alpha f](\lambda) = i\lambda \mathcal{F}_\alpha[f](\lambda)$;
- The Dunkl transform leaves $\mathcal{S}(\mathbb{R})$ invariant.

Bewijs. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$\frac{d^n}{d\lambda^n}\mathcal{F}_\alpha[f](\lambda) = \int_{\mathbb{R}} \frac{d^n}{d\lambda^n} E_\alpha(-x,\lambda) f(x) d\mu_\alpha(x)$$

for all $n \in \mathbb{N}$, thus

$$\sup_{\lambda\in\mathbb{R}} \left| \frac{d^n}{d\lambda^n}\mathcal{F}_\alpha[f](\lambda) \right| \leq \int_{\mathbb{R}} |x^n f(x)|\, d\mu_\alpha(x) < +\infty.$$

Using Proposition 2.47 we obtain

$$D_{\alpha,\lambda}\mathcal{F}_\alpha[f](\lambda) = \int_{\mathbb{R}} D_{\alpha,\lambda} E_\alpha(-x,\lambda) f(x) d\mu_\alpha(x)$$
$$= \int_{\mathbb{R}} E_\alpha(-x,\lambda)(-ix) f(x) d\mu_\alpha(x) = -\mathcal{F}_\alpha[ixf](\lambda)$$

and

$$\mathcal{F}_\alpha[D_{\alpha,x}f](\lambda) = \int_{\mathbb{R}} E_\alpha(-x,\lambda) D_{\alpha,x} f(x) d\mu_\alpha(x)$$
$$= -\int_{\mathbb{R}} D_{\alpha,x} E_\alpha(-x,\lambda) f(x) d\mu_\alpha(x) = (i\lambda)\mathcal{F}_\alpha[f](\lambda).$$

To prove the last property, notice that it suffices to prove that $\frac{d^n}{d\lambda^n}(\lambda^m \mathcal{F}_\alpha[f](\lambda))$ is bounded for arbitrary $m, n \in \mathbb{N}$. Applying previous property m times and Corollary 2.48 we have

$$\begin{aligned}\lambda^m \mathcal{F}_\alpha[f](\lambda) &= \frac{1}{(-i)^m}\int_{\mathbb{R}} (-i\lambda)^m E_\alpha(-x,\lambda) f(x) d\mu_\alpha(x) \\ &= \frac{1}{(-i)^m}\int_{\mathbb{R}} D^m_{\alpha,x} E_\alpha(-x,\lambda) f(x) d\mu_\alpha(x) \\ &= \frac{(-1)^m}{(-i)^m}\int_{\mathbb{R}} E_\alpha(-x,\lambda) D^m_{\alpha,x} f(x) d\mu_\alpha(x).\end{aligned}$$

Consequently, we obtain

$$\frac{d^n}{d\lambda^n}(\lambda^m \mathcal{F}_\alpha[f](\lambda)) = \frac{1}{i^m}\int_{\mathbb{R}} \frac{d^n}{d\lambda^n} E_\alpha(-x,\lambda) D^m_{\alpha,x} f(x) d\mu_\alpha(x)$$

and

$$\sup_{\lambda\in\mathbb{R}} \left| \frac{d^n}{d\lambda^n}(\lambda^m \mathcal{F}_\alpha[f](\lambda)) \right| \leq \int_{\mathbb{R}} |x^n D^m_{\alpha,x} f(x)| d\mu_\alpha(x)$$
$$\leq \frac{C}{2^{\alpha+1}\Gamma(\alpha+1)} \sup_{x\in\mathbb{R}} |(1+|x|)^{n+k} D^m_{\alpha,x} f(x)| < +\infty \quad (2.28)$$

for all $m, n \in \mathbb{N}$, where

$$C = \int_{\mathbb{R}} \frac{|x|^{2\alpha+1}}{(1+|x|)^k} dx < +\infty, \quad k > 2(\alpha+1) \quad \text{and} \quad k \in \mathbb{N}. \quad (2.29)$$

Here we have used (2.10). □

Proposition 2.49. *The Dunkl transform is a linear continuous map on* $\mathcal{S}(\mathbb{R})$.

Bewijs. As the Dunkl transform is a linear map, so we prove only its continuity. Let $f_j \to f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$. Then using (2.28) and (2.9) we obtain

$$\begin{aligned}\sup_{\lambda\in\mathbb{R}}\left|\frac{d^n}{d\lambda^n}\left(\lambda^m\left(\mathcal{F}_\alpha[f_j]-\mathcal{F}_\alpha[f]\right)(\lambda)\right)\right| &= \sup_{\lambda\in\mathbb{R}}\left|\frac{d^n}{d\lambda^n}\left(\lambda^m\mathcal{F}_\alpha[f_j-f](\lambda)\right)\right| \\ &\leq \int_{\mathbb{R}}\left|x^n D^m_{\alpha,x}(f_j-f)(x)\right| d\mu_\alpha(x) \\ &\leq \frac{C}{2^{\alpha+1}\Gamma(\alpha+1)}\sup_{x\in\mathbb{R}}\left|(1+|x|)^{n+k}D^m_{\alpha,x}(f_j-f)(x)\right| \\ &\to 0,\end{aligned}$$

where C is constant defined by (2.29). □

Lemma 2.50 (Multiplication formula for the Dunkl transform). *Let $f, g \in L^1(\mathbb{R}, d\mu_\alpha)$. Then*

$$\int_{\mathbb{R}}\mathcal{F}_\alpha[f](\lambda)g(\lambda)d\mu_\alpha(\lambda) = \int_{\mathbb{R}} f(\lambda)\mathcal{F}_\alpha[g](\lambda)d\mu_\alpha(\lambda).$$

Bewijs. Applying Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{F}_\alpha[f](\lambda)g(\lambda)d\mu_\alpha(\lambda) &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}E_\alpha(-x,\lambda)f(x)d\mu_\alpha(x)\right]g(\lambda)d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}E_\alpha(-x,\lambda)g(\lambda)d\mu_\alpha(\lambda)\right]f(x)d\mu_\alpha(x) \\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}}E_\alpha(-\lambda,x)g(\lambda)d\mu_\alpha(\lambda)\right]f(x)d\mu_\alpha(x) \\ &= \int_{\mathbb{R}}\mathcal{F}_\alpha[g](x)f(x)d\mu_\alpha(x) \\ &= \int_{\mathbb{R}}\mathcal{F}_\alpha[g](\lambda)f(\lambda)d\mu_\alpha(\lambda),\end{aligned}$$

where we have used the Dunkl kernel's property. □

Let $\alpha \geq -\frac{1}{2}$. Note that the Dunkl operator D_α is skew symmetric with respect to the L^2-norm associated to the measure μ_α, i.e.

$$(D_\alpha f, g)_{2,\alpha} = \int_{\mathbb{R}} D_\alpha f(x)\overline{g(x)}d\mu_\alpha(x) = -\int_{\mathbb{R}} f(x)\overline{D_\alpha g(x)}d\mu_\alpha(x) = (f, D^*_\alpha g)_{2,\alpha},$$

where $f, g \in C^\infty(\mathbb{R})$.

Now, we show that the Dunkl transform can be extended from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ by duality (see [77]).

Definition 2.51 (The Dunkl transform of tempered distributions). If $u \in \mathcal{S}'(\mathbb{R})$, we can define its (generalised) Dunkl transform by setting

$$\langle\mathcal{F}_\alpha[u], \varphi\rangle := \langle u, \mathcal{F}_\alpha[\varphi]\rangle \tag{2.30}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$.

We can interpret functions in $L^p(\mathbb{R}, d\mu_\alpha)$, $1 \le p < +\infty$, as tempered distributions. If $f \in L^p(\mathbb{R}, d\mu_\alpha)$, we define functional u_f by

$$\langle u_f, \varphi\rangle := \int_{\mathbb{R}} f(x)\varphi(x)d\mu_\alpha(x),$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. By Hölder's inequality, we define that

$$|\langle u_f, \varphi\rangle| \le \|f\|_{p,\alpha}\|\varphi\|_{q,\alpha},$$

for $1/p + 1/q = 1$. Hence, $\langle u_f, \varphi\rangle$ is well defined in view of the simple inclusion $\mathcal{S}(\mathbb{R}) \subset L^q(\mathbb{R}, d\mu_\alpha)$, for all $1 \le p \le +\infty$ (see Lemma 2.42). Therefore,

$$\langle \mathcal{F}_\alpha[u_f], \varphi\rangle = \langle u_f, \mathcal{F}_\alpha[\varphi]\rangle = \int_{\mathbb{R}} f(x)\mathcal{F}_\alpha[\varphi](x)d\mu_\alpha(x).$$

Then using Lemma 2.50, we obtain

$$\int_{\mathbb{R}} f(x)\mathcal{F}_\alpha[\varphi](x)d\mu_\alpha(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha[f](x)\varphi(x)d\mu_\alpha(x) = \langle u_{\mathcal{F}_\alpha[f]}, \varphi\rangle.$$

Hence, we have

$$\langle \mathcal{F}_\alpha[u_f], \varphi\rangle = \langle u_{\mathcal{F}_\alpha[f]}, \varphi\rangle.$$

Proposition 2.52 (**The Dunkl transform on $\mathcal{S}'(\mathbb{R})$**). *The Dunkl transform is a continuous linear operator from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$.*

Bewijs. Since the Dunkl transform leaves invariant the Schwartz space $\mathcal{S}(\mathbb{R})$, (2.30) is well defined.

Let $\varphi, \psi \in \mathcal{S}(\mathbb{R})$, $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R})$ and $\lambda, \xi \in \mathbb{K}$. Then we have

$$\begin{aligned}\langle \mathcal{F}_\alpha[u], \lambda\varphi + \xi\psi\rangle &= \langle u, \mathcal{F}_\alpha[\lambda\varphi + \xi\psi]\rangle = \langle u, \lambda\mathcal{F}_\alpha[\varphi] + \xi\mathcal{F}_\alpha[\psi]\rangle \\ &= \lambda\langle u, \mathcal{F}_\alpha[\varphi]\rangle + \xi\langle u, \mathcal{F}_\alpha[\psi]\rangle = \lambda\langle \mathcal{F}_\alpha[u], \varphi\rangle + \xi\langle \mathcal{F}_\alpha[u], \psi\rangle\end{aligned}$$

and

$$\langle \mathcal{F}_\alpha[u], \varphi_j\rangle = \langle u, \mathcal{F}_\alpha[\varphi_j]\rangle \to \langle u, \mathcal{F}_\alpha[\varphi]\rangle = \langle \mathcal{F}_\alpha[u], \varphi\rangle$$

since $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R})$ and the Dunkl transform is a linear continuous map on $\mathcal{S}(\mathbb{R})$ (Proposition 2.49). Thus, $\mathcal{F}_\alpha[u] \in \mathcal{S}'(\mathbb{R})$ defined by (2.30). Now, it follows that it is also continuous as a mapping from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$, i.e. if $u_j \to u$ in $\mathcal{S}'(\mathbb{R})$, then

$$\langle \mathcal{F}_\alpha[u_j], \varphi\rangle = \langle u_j, \mathcal{F}_\alpha[\varphi]\rangle \to \langle u, \mathcal{F}_\alpha[\varphi]\rangle = \langle \mathcal{F}_\alpha[u], \varphi\rangle,$$

which means that $\mathcal{F}_\alpha[u_j] \to \mathcal{F}_\alpha[u]$ in $\mathcal{S}'(\mathbb{R})$. □

The following properties holds (see [77, p. 10]):

- $\mathcal{F}_\alpha$ is a topological isomorphism of $\mathcal{S}'(\mathbb{R})$ onto itself;
- $\mathcal{F}_\alpha[D_\alpha u](\lambda) = i\lambda\mathcal{F}_\alpha[u](\lambda)$, for all $u \in \mathcal{S}'(\mathbb{R})$;
- $D_\alpha\mathcal{F}_\alpha[u] = -\mathcal{F}_\alpha[ixu]$, for all $u \in \mathcal{S}'(\mathbb{R})$.

We also define the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ on $\mathcal{S}'(\mathbb{R})$ using (2.30), i.e.

$$\langle \mathcal{F}_\alpha^{-1}[u], \varphi\rangle := \langle u, \mathcal{F}_\alpha^{-1}[\varphi]\rangle$$

Theorem 2.53 (**Fourier inversion formula for tempered distributions**). *Operators $\mathcal{F}_\alpha$ and $\mathcal{F}_\alpha^{-1}$ are inverse to each other on $\mathcal{S}'(\mathbb{R})$, i.e.,*

$$\mathcal{F}_\alpha\mathcal{F}_\alpha^{-1} = \mathcal{F}_\alpha^{-1}\mathcal{F}_\alpha = identity \quad on \quad \mathcal{S}'(\mathbb{R}).$$

Bewijs. Let $u \in \mathcal{S}'(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$. Then we have

$$\langle \mathcal{F}_\alpha \mathcal{F}_\alpha^{-1}[u], \varphi \rangle = \langle \mathcal{F}_\alpha^{-1}[u], \mathcal{F}_\alpha[\varphi] \rangle = \langle u, \mathcal{F}_\alpha \mathcal{F}_\alpha^{-1}[\varphi] \rangle = \langle u, \varphi \rangle.$$

Same calculation can be used for second equation. □

2.2.2. *The Dunkl convolution.* We have the following product formula for the function $j_\alpha(x\lambda)$ with $\alpha > -\frac{1}{2}$ and parameter $\lambda \in \mathbb{C}$ ([93, Formula 1.II.23, p. 13]):

$$j_\alpha(x\lambda) j_\alpha(y\lambda) = \int_0^{+\infty} j_\alpha(z\lambda) k_\alpha(x, y, z) z^{2\alpha+1} dz$$

for $x, y > 0$, where

$$k_\alpha(x, y, z) = 2^{2\alpha-1} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} \frac{\Delta(x, y, z)^{2\alpha-1}}{(xyz)^{2\alpha}} \cdot 1_{[|x-y|, x+y]}(z). \tag{2.31}$$

Here 1_A is the indicator function of A and

$$\Delta(x, y, z) := \frac{1}{4}\sqrt{(x+y+z)(x+y-z)(x-y+z)(y+z-x)}$$

denotes the area of the triangle with sides $x, y, z > 0$. The function $k_\alpha(x, y, z)$ satisfies the following properties ([93, p. 13-14]):

- For all $z > 0$, $k_\alpha(x, y, z) \geq 0$,
- We have for $x, y > 0$:
$$\int_0^{+\infty} k_\alpha(x, y, z) z^{2\alpha+1} dz = 1,$$
- We have for all $x, y, z > 0$:
$$k_\alpha(x, y, z) = k_\alpha(y, x, z) \quad \text{and} \quad k_\alpha(x, y, z) = k_\alpha(x, z, y).$$

For our convenience, we fix some notations. For all $x, y, z \in \mathbb{R}$, we put

$$b_{x,y,z} := \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if} \quad x, y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\rho(x, y, z) := \frac{1}{2}(1 - b_{x,y,z} + b_{z,x,y} + b_{z,y,x}).$$

Theorem 2.54. [75, Theorem 2.4, p. 5] *(1) Let $\alpha > -\frac{1}{2}$ and $\lambda \in \mathbb{C}$. Then the Dunkl kernel E_α satisfies the following product formula:*

$$E_\alpha(x, \lambda) E_\alpha(y, \lambda) = \int_{\mathbb{R}} E_\alpha(z, \lambda) d\nu_{x,y}(z) \tag{2.32}$$

for $x, y \in \mathbb{R}$, where

$$d\nu_{x,y}(z) := \begin{cases} W_\alpha(x, y, z)|z|^{2\alpha+1} dz & \textit{if} \quad x, y \neq 0, \\ d\delta_x(z) & \textit{if} \quad y = 0, \\ d\delta_y(z) & \textit{if} \quad x = 0. \end{cases}$$

with kernel

$$W_\alpha(x, y, z) = k_\alpha(|x|, |y|, |z|)\rho(x, y, z),$$

where k_α is the Bessel kernel (2.31).
(2) The measures $\nu_{x,y}$ have the following properties:

- *$supp\nu_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$ for $x, y \neq 0$,*
- *$\|\nu_{x,y}\| := \int_{\mathbb{R}} W_\alpha(x,y,z)|z|^{2\alpha+1}dz \leq 4$ for all $x, y \in \mathbb{R}$.*

Remark 2.55. In Theorem 2.54, δ_x is the Dirac measure. So, we have

- If $y = 0$, then
$$E_\alpha(x,\lambda) = E_\alpha(x,\lambda)E_\alpha(0,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda)d\delta_x(z) = E_\alpha(x,\lambda),$$
- If $x = 0$, then
$$E_\alpha(y,\lambda) = E_\alpha(0,\lambda)E_\alpha(y,\lambda) = \int_{\mathbb{R}} E_\alpha(z,\lambda)d\delta_y(z) = E_\alpha(y,\lambda).$$

Remark 2.56. Let $x, y \neq 0$. Then from
$$\begin{aligned}E_\alpha(x,\lambda)E_\alpha(y,\lambda) &= \int_{\mathbb{R}} E_\alpha(z,\lambda)W_\alpha(x,y,z)|z|^{2\alpha+1}dz \\ &= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(z,\lambda)W_\alpha(x,y,z)d\mu_\alpha(z)\end{aligned}$$
we obtain
$$W_\alpha(x,y,z) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}} E_\alpha(-z,\lambda)E_\alpha(x,\lambda)E_\alpha(y,\lambda)d\mu_\alpha(\lambda). \tag{2.33}$$

Lemma 2.57. *Let $x, y, z \in \mathbb{R}$. Then*
$$W_\alpha(x,-y,z) = W_\alpha(x,-z,y).$$
Furthermore, we have
$$W_\alpha(x,-y,z)|z|^{2\alpha+1}dzd\mu_\alpha(y) = W_\alpha(x,-z,y)|y|^{2\alpha+1}dyd\mu_\alpha(z).$$

Bewijs. For any $x, y, z \in \mathbb{R}$ a short calculation gives us the following equalities
$$\begin{aligned}b_{x,-y,z} &= \frac{x^2+(-y)^2-z^2}{2x(-y)} = -\frac{x^2+y^2-z^2}{2xy} \\ b_{z,x,-y} &= \frac{z^2+x^2-(-y)^2}{2zx} = \frac{z^2+x^2-y^2}{2zx} \\ b_{z,-y,x} &= \frac{z^2+(-y)^2-x^2}{2z(-y)} = -\frac{z^2+y^2-x^2}{2zy}\end{aligned}$$
and
$$\begin{aligned}\rho(x,-y,z) &= \frac{1}{2}(1-b_{x,-y,z}+b_{z,x,-y}+b_{z,-y,x}) \\ &= \frac{1}{2}\left(1+\frac{x^2+y^2-z^2}{2xy}+\frac{z^2+x^2-y^2}{2zx}-\frac{z^2+y^2-x^2}{2zy}\right) \\ &= \frac{1}{2}\left(1+\frac{x^2+z^2-y^2}{2zx}+\frac{y^2+x^2-z^2}{2xy}-\frac{y^2+z^2-x^2}{2zy}\right) \\ &= \frac{1}{2}(1-b_{x,-z,y}+b_{y,x,-z}+b_{y,-z,x})\end{aligned}$$

$$= \rho(x, -z, y).$$

Then using property of the function $k_\alpha(x, y, z)$ we obtain

$$\begin{aligned} W_\alpha(x, -y, z) &= k_\alpha(|x|, |-y|, |z|)\rho(x, -y, z) \\ &= k_\alpha(|x|, |-z|, |y|)\rho(x, -z, y) \\ &= W_\alpha(x, -z, y). \end{aligned}$$

Thus, we have

$$\begin{aligned} W_\alpha(x, -y, z)|z|^{2\alpha+1} dz d\mu_\alpha(y) &= W_\alpha(x, -z, y) \frac{|y|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)} dy |z|^{2\alpha+1} dz \\ &= W_\alpha(x, -z, y)|y|^{2\alpha+1} dy d\mu_\alpha(z). \end{aligned}$$

□

For all $x, y \in \mathbb{R}$ and f a continuous function on $\mathbb{R}$, we define

$$\tau_x f(y) := \int_{\mathbb{R}} f(z) d\nu_{x,y}(z). \tag{2.34}$$

The operators $\tau_x, x \in \mathbb{R}$ are called **Dunkl translation operators** on $\mathbb{R}$.

Proposition 2.58. [88, Proposition 2, p. 20] *The operators $\tau_x, x \in \mathbb{R}$ have the following properties:*

- *for all $x \in \mathbb{R}$ and $f \in L^p(\mathbb{R}, d\mu_\alpha)$, $p \in [1, +\infty]$, we have*

$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha},$$

- *for all $\lambda, x \in \mathbb{R}$ and $f \in L^1(\mathbb{R}, d\mu_\alpha)$, we obtain*

$$\mathcal{F}_\alpha[\tau_x f](\lambda) = E_\alpha(x, \lambda)\mathcal{F}_\alpha[f](\lambda).$$

For two continuous functions f and g on $\mathbb{R}$ with compact supports, we define a convolution product $*_\alpha$ by

$$(f *_\alpha g)(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y) d\mu_\alpha(y), \quad x \in \mathbb{R},$$

where $\tau_x, x \in \mathbb{R}$ is the Dunkl translation operator on $\mathbb{R}$.

Remark 2.59. Note that $*_{-\frac{1}{2}}$ is the standard convolution $*$.

Proposition 2.60. [88, Proposition 3, p. 21] *(i) Let $p, q, r \in [1, \infty]$ and satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then the map $(f, g) \mapsto f *_\alpha g$ can be extended to a continuous map from $L^p(\mathbb{R}, d\mu_\alpha) \times L^q(\mathbb{R}, d\mu_\alpha)$ to $L^r(\mathbb{R}, d\mu_\alpha)$, and*

$$\|f *_\alpha g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

(ii) For any $f \in L^1(\mathbb{R}, d\mu_\alpha)$ and $g \in L^2(\mathbb{R}, d\mu_\alpha)$, we have

$$\mathcal{F}_\alpha[f *_\alpha g](\lambda) = \mathcal{F}_\alpha[f](\lambda)\mathcal{F}_\alpha[g](\lambda), \quad \lambda \in \mathbb{R}.$$

2.2.3. *The generalized Taylor formula.* A *polynomial* is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n,$$

where $a_0, ..., a_n$ are constants. Polynomials are convenient to work with because their values can be calculated easily. Therefore, they have been extensively used to approximate more complicated functions. Taylor's theorem is one of the oldest and most important results on this question.

From basic theory of calculus it is known that, if we consider the function given by the sum of a power series with radius of convergence R (R may be $+\infty$)

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

then the power series has the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k, \quad |x| < R, \tag{2.35}$$

where all derivatives $f^{(k)}(0)$ should exist.

Now, question is what if we use the Dunkl operator instead of usual derivative, as natural generalization of usual first order derivative. In which case, we obtain more general form than (2.35), as stated in Lemma 2.61.

Lemma 2.61. *Let $\alpha \geq -\frac{1}{2}$. Suppose that f is an analytic function*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

on $\mathbb{R}$ with radius of convergence $R > 0$ [R may be $+\infty$]. Then the original power series has the form

$$f(x) = \sum_{k=0}^{\infty} \frac{D_\alpha^k f(0)}{\gamma_\alpha(k)} x^k, \quad |x| < R. \tag{2.36}$$

Bewijs. Taking into account the fact

$$D_\alpha^n(x^k) = \begin{cases} 0, & \text{if} \quad k < n, \\ \gamma_\alpha(n), & \text{if} \quad k = n, \\ P_\alpha(n; x), & \text{if} \quad k > n, \end{cases} \tag{2.37}$$

which we will prove later, and applying the Dunkl operator to the function n times we obtain

$$D_\alpha^n f(x) = \sum_{k=0}^{\infty} a_k D_\alpha^n(x^k) = a_n \gamma_\alpha(n) + P_\alpha(n; x),$$

where the polynomial $P_\alpha(n; x)$ has a property $P_\alpha(n; 0) = 0$. Thus $D_\alpha^n f(0) = a_n \gamma_\alpha(n)$, which proves the (2.36). Now let us prove the (2.37). In (2.37), the second one is not obvious, so we will focus on that one. To prove $D_\alpha^n(x^n) = \gamma_\alpha(n)$ we use mathematical induction. Let $n = 0$ and $n = 1$, then $D_\alpha^0(x^0) = x^0 = 1 = \gamma_\alpha(0)$ and $D_\alpha(x) = 1 + (\alpha + 1/2)\, 2 = 2(\alpha + 1) = \gamma_\alpha(1)$, respectively. After we suppose $D_\alpha^k(x^k) = \gamma_\alpha(k)$ is true. Then we have

$$D_\alpha^{k+1}(x^{k+1}) = D_\alpha^k(D_\alpha x^{k+1}) = D_\alpha^k\left((k+1)x^k + \left(\alpha+\frac{1}{2}\right)\frac{x^{k+1}-(-x)^{k+1}}{x}\right)$$
$$= \left(k+1+\left(\alpha+\frac{1}{2}\right)(1+(-1)^k)\right)D_\alpha^k x^k = (k+1+(2\alpha+1)\theta_{k+1})\,\gamma_\alpha(k).$$

Finally, using recurrent formula for γ_α ([68, Formulas 2.2.7, p. 372]) we obtain $D_\alpha^{k+1}(x^{k+1}) = \gamma_\alpha(k+1)$, which completes our proof. □

Now, we assume that $a > 0$ and f is defined on $(-a, a)$ and the N^{th} derivative $f^{(N)}$ exists on $(-a, a)$. Then we can define N^{th} -order Taylor polynomial for f about 0, i.e.

$$T_N(x) = \sum_{k=0}^{N}\frac{D_\alpha^k f(0)}{\gamma_\alpha(k)}x^k$$

and for $N \geq 1$, remainder $R_{N+1}(x; f; 0)$ is defined by

$$R_{N+1}(x; f; 0) = f(x) - T_N(x). \tag{2.38}$$

So, from (2.38) we obtain

$$f(x) = \sum_{k=0}^{\infty}\frac{D_\alpha^k f(0)}{\gamma_\alpha(k)}x^k \quad \text{if and only if} \quad \lim_{N\to\infty} R_{N+1}(x; f; 0) = 0.$$

The next question is, does such a remainder exist? Classical Taylor's Theorem states that such remainder exist, for $\alpha = -1/2$, i.e.

Theorem 2.62. *We assume that f is defined on (a, b), where $a < c < b$, and the (N+1)th derivative $f^{(N+1)}$ exists on (a, b). Then for each $x \neq c$ in (a, b) there is some y between c and x such that*

$$R_{N+1}(x; f; c) = \frac{f^{(N+1)}f(y)}{(N+1)!}(x-c)^{N+1}.$$

Corollary 2.63. *We suppose that f is defined on (a, b), where $a < c < b$, and the (N+1)th derivative $f^{(N+1)}$ exists on (a, b) and are bounded by a single constant C. Then*

$$\lim_{N\to\infty} R_{N+1}(x; f; c) = 0,$$

for all $x \in (a, b)$.

So, can we extend this results in a case $\alpha > -1/2$. This extension was obtained by Mohamed Ali Mourou [59] in 2003. He extended Theorem 2.62 and Corollary 2.63 to a first-order general differential-difference operator

$$\Lambda f = \frac{df}{dx} + \frac{A'(x)}{A(x)}\frac{f(x) - f(-x)}{2},$$

on the real line which in the particular case, when $A(x) = x^{2\alpha+1}$, $\alpha \geq -1/2$, gives the Dunkl operator D_α. He established a generalized Taylor formula with integral remainder. Before, let us introduce some notations from [59].

Let S be the subset of $\mathbb{R}^2$ defined by $S = \{(x,y) \in \mathbb{R}^2 : 0 < |y| \le |x|\}$. Then using recursive integral formulae

$$u_0(x,y) = \frac{sgn(x)}{2|x|^{2\alpha+1}}, \quad v_0(x,y) = \frac{sgn(y)}{2|y|^{2\alpha+1}}$$

and

$$u_{k+1}(x,y) = \int_{|y|}^{|x|} v_k(x,z)dz, \quad v_{k+1}(x,y) = \frac{sgn(y)}{|y|^{2\alpha+1}} \int_{|y|}^{|x|} u_k(x,z)|z|^{2\alpha+1}dz$$

we define sequences of functions $\{u_k(x,y)\}$, $\{v_k(x,y)\}$, $k \in \mathbb{N}$ on S. The central result of the paper [59] is stated as following.

Theorem 2.64. *Let $f \in C^\infty(\mathbb{R})$. Then we can obtain the generalized Taylor formula for the Dunkl operator D_α with integral remainder:*

$$f(x) = \sum_{k=0}^{N} \frac{D_\alpha^k f(0)}{\gamma_\alpha(k)} x^k + \int_{-|x|}^{|x|} w_N(x,y) D_\alpha^{N+1} f(y)|y|^{2\alpha+1}dy$$

for all $N \in \mathbb{N}$, where

$$w_N(x,y) = u_N(x,y) + v_N(x,y).$$

Theorem 2.65. *Let $f \in C^\infty(\mathbb{R})$. Assume that there are $M, \rho > 0$ such that*

$$\sup_{|x|\le\rho} |D_\alpha^n f(x)| \le M^{n+1} n!,$$

for all $n \in \mathbb{N}$. Then there exists an $r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{D_\alpha^n f(0)}{\gamma_\alpha(n)} x^n,$$

for $|x| \le r$. Moreover, the series converges uniformly for $|x| \le r$.

2.3. **Fractional differential operators.** In this section we give the definitions of the Riemann-Liouville fractional integrals and fractional derivatives, Caputo and bi-ordinal Hilfer fractional derivatives on a finite interval of the real line and present some of their properties. Also, we here present some necessary information about Mittag-Leffler functions.

2.3.1. *Riemann-Liouville and Caputo fractional operators.* Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$ and $-\infty < a < b < \infty$.

Definition 2.66. [48, p. 69, formulas (2.1.1) and (2.1.2)] The left-hand sided $I_{a+}^\gamma f$ and the right-hand sided $I_{b-}^\gamma f$ **Riemann-Liouville fractional integrals** of order $\gamma > 0$ are defined by

$$I_{a+}^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\gamma}}, \quad t \in (a,b],$$

and

$$I_{b-}^\gamma f(t) := \frac{1}{\Gamma(\gamma)} \int_t^b \frac{f(\tau)d\tau}{(\tau-t)^{1-\gamma}}, \quad t \in [a,b),$$

respectively. Here Γ is Euler's gamma function.

Definition 2.67. [48, p. 70, formulas (2.1.5) and (2.1.6)] The left-hand sided $D^{\gamma}_{a+}f$ and the right-hand sided $D^{\gamma}_{b-}f$ **Riemann-Liouville fractional derivatives** of order γ, which are expressed by

$$D^{\gamma}_{a+}f(t) := \left(\frac{d}{dt}\right)^n I^{n-\gamma}_{a+}f(t), \quad \forall t \in (a,b],$$

and

$$D^{\gamma}_{b-}f(t) := (-1)^n \left(\frac{d}{dt}\right)^n I^{n-\gamma}_{b-}f(t), \quad \forall t \in [a,b),$$

respectively. Here $n = [\gamma] + 1$.

The Riemann-Liouville fractional integrals and derivatives have the following properties:

- If $\gamma > 0$ and $\beta > 0$, then ([48, p. 71, Property 2.1])
$$\left(I^{\gamma}_{a+}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}(x-a)^{\beta+\gamma-1}$$
and
$$\left(I^{\gamma}_{b-}(b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}(b-x)^{\beta+\gamma-1};$$
- Let $\gamma_1 > 0$, $\gamma_2 > 0$, $t \in [a,b]$, and $f \in L_p(a,b)$, $1 \le p < +\infty$. Then we have ([48, p. 73, Lemma 2.3])
$$I^{\gamma_1}_{a+}I^{\gamma_2}_{a+}f(t) = I^{\gamma_1+\gamma_2}_{a+}f(t) \quad \text{and} \quad I^{\gamma_1}_{b-}I^{\gamma_2}_{b-}f(t) = I^{\gamma_1+\gamma_2}_{b-}f(t);$$
- Let $\gamma > 0$ and $n = [\gamma] + 1$. If $f \in L_1(a,b)$ and $I^{n-\gamma}_{a+}f, I^{n-\gamma}_{b-}f \in AC[a,b]$, then the equalities
$$I^{\gamma}_{a+}D^{\gamma}_{a+}f(t) = f(t) - \sum_{j=1}^{n} \frac{(t-a)^{\gamma-j}}{\Gamma(\gamma-j+1)} \left[\lim_{t\to a+} \left(\frac{d}{dt}\right)^{n-j} I^{n-\gamma}_{a+}f(t)\right]$$
and
$$I^{\gamma}_{b-}D^{\gamma}_{b-}f(t) = f(t) - \sum_{j=1}^{n} \frac{(-1)^{n-j}(b-t)^{\gamma-j}}{\Gamma(\gamma-j+1)} \left[\lim_{t\to b-} \left(\frac{d}{dt}\right)^{n-j} I^{n-\gamma}_{b-}f(t)\right]$$
hod almost everywhere on $[a,b]$ ([48, p. 74-75, Lemma 2.5-2.6]).

Definition 2.68. [48, p. 91, formulas (2.4.1) and (2.4.2)] The left-hand sided $\mathcal{D}^{\gamma}_{a+}f$ and the right-hand sided $\mathcal{D}^{\gamma}_{b-}f$ **Caputo fractional derivatives** of order γ ($\gamma \ge 0$), which are defined by the formulas

$$\mathcal{D}^{\gamma}_{a+}f(t) := D^{\gamma}_{a+}\left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k\right], \quad t \in (a,b],$$

and

$$\mathcal{D}^{\gamma}_{b-}f(t) := D^{\gamma}_{b-}\left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-t)^k\right], \quad t \in [a,b),$$

respectively. Here

$$n = \begin{cases} [\gamma]+1, & \text{if} \quad \gamma \notin \mathbb{N}_0. \\ \gamma, & \text{if} \quad \gamma \in \mathbb{N}_0. \end{cases}$$

Definition 2.69. [25, p. 18, Definition 3] Let X be a Banach space. We say that $u \in C^\gamma([0,T],X)$ if $u \in C([0,T],X)$ and $\mathcal{D}^\gamma_{a+}u \in C([0,T],X)$.

Remark 2.70. [48, p. 92, Theorem 2.1] Assume that $\gamma \geq 0$ and $f \in AC^n[a,b]$. Then Caputo fractional derivative exists almost everywhere on $[a,b]$. Moreover, the left-sided and right-sided Caputo fractional derivatives are represented by

$$\mathcal{D}^\gamma_{a+}f(t) := I^{n-\gamma}_{a+}\left(\frac{d}{dt}\right)^n f(t), \quad t \in (a,b],$$

and

$$\mathcal{D}^\gamma_{b-}f(t) := (-1)^n I^{n-\gamma}_{b-}\left(\frac{d}{dt}\right)^n f(t), \quad t \in [a,b),$$

respectively. Here $n = [\gamma]+1$.

2.3.2. *Bi-ordinal Hilfer fractional derivatives.* The bi-ordinal Hilfer fractional derivative

$$D^{\gamma_1,\gamma_2}_{a\pm}f(x) = \left(\pm I^{\gamma_2(1-\gamma_1)}_{a\pm}\frac{d}{dx}\left(I^{(1-\gamma_2)(1-\gamma_1)}_{a\pm}f\right)\right)(x),$$

where $0 < \gamma_1 < 1$, and $0 \leq \gamma_2 \leq 1$, is introduced by R. Hilfer in [39, p. 113, Definition 3.3] in 2000, as a new generalization of the Riemann-Liouville derivative. A few applications of this operator were investigated by numerous mathematicians, for example, [40, 41]. A generalization of this operator

$$D^{(\gamma_1,\gamma_2)s}_{a+}f(t) = I^{s(n-\gamma_1)}_{a+}\left(\frac{d}{dt}\right)^n I^{(1-s)(n-\gamma_2)}_{a+}f(t),$$

where $n-1 < \gamma_1 \leq n$, $n-1 < \gamma_2 \leq n$, and $0 \leq s \leq 1$, is introduced in Toshtemirov's PhD dissertatin [91]. In the PhD dissertation in chapter IV, the author considered direct and inverse problems for the pseudo-parabolic equation with the bi-ordinal Hilfer fractional derivative.

Definition 2.71. The left-hand sided and right-hand sided **bi-ordinal Hilfer fractional derivatives** ([91, p. 16, Definition 1.3.11]) of orders γ_1 $(n-1 < \gamma_1 \leq n)$ and γ_2 $(n-1 < \gamma_2 \leq n)$ type $s \in [0,1]$ are expressed by

$$D^{(\gamma_1,\gamma_2)s}_{a+}f(t) := I^{s(n-\gamma_1)}_{a+}\left(\frac{d}{dt}\right)^n I^{(1-s)(n-\gamma_2)}_{a+}f(t), \quad t \in (a,b],$$

and

$$D^{(\gamma_1,\gamma_2)s}_{b-}f(t) := I^{s(n-\gamma_1)}_{b-}\left(-\frac{d}{dt}\right)^n I^{(1-s)(n-\gamma_2)}_{b-}f(t), \quad t \in [a,b),$$

respectively.

Remark 2.72. In the case $s = 0$, we obtain the classical Riemann-Liouville fractional derivatives and when $s = 1$, they turn into the Caputo fractional derivatives (Remark 2.70).

Remark 2.73. The left-sided bi-ordinal Hilfer fractional derivative $D_{a+}^{(\gamma_1,\gamma_2)s} f$ can be rewritten as following

$$\begin{aligned} D_{a+}^{(\gamma_1,\gamma_2)s} f(t) = I_{a+}^{s(n-\gamma_1)} \left(\frac{d}{dt}\right)^n I_{a+}^{(1-s)(n-\gamma_2)} f(t) = I_{a+}^{s(n-\gamma_1)} \left(\frac{d}{dt}\right)^n I_{a+}^{n-\eta} f(t) \\ = I_{a+}^{s(n-\gamma_1)} D_{a+}^{\eta} f(t) = I_{a+}^{\eta-\delta} D_{a+}^{\eta} f(t), \end{aligned}$$

for $a \leq t \leq b$, where $\eta = \gamma_2 + s(n-\gamma_2)$ and $\delta = \gamma_2 + s(\gamma_1 - \gamma_2)$, which have the properties $n-1 < \eta, \delta \leq n$, and $\gamma_2 < \eta$, $n \in \mathbb{N}$.

2.3.3. *Mittag-Leffler functions.* Here we present the definitions and some properties of classical Mittag-Leffler functions. More detailed information may be found in the book [48, p. 40–42].

The **Mittag-Leffler function** $\mathbb{E}_{\gamma_1,\gamma_2}$ is a special function, a complex function which depends on $\gamma_1, \gamma_2 \in \mathbb{C}$. It may be defined by the following series when the real part of γ_1 is strictly positive

$$\mathbb{E}_{\gamma_1,\gamma_2}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma_1 k + \gamma_2)},$$

where Γ is the gamma function. When $\gamma_2 = 1$, we obtain special case of the Mittag-Leffler function defined by

$$\mathbb{E}_{\gamma_1,1}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma_1 k + 1)}.$$

Here we able to see that $\mathbb{E}_{\gamma_1,1}(0) = 1$. In particular case, when $\gamma_1 = \gamma_2 = 1$, we have

$$\mathbb{E}_{1,1}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{+\infty} \frac{z^k}{k!} = \exp(z).$$

For $0 < \gamma_1 < 1$ (not true for $\gamma_1 \geq 1$) we have the following estimates ([81, Theorem 4, p. 21]) for Mittag-Leffler function

$$\frac{1}{1+\Gamma(1-\gamma_1)x} \leq \mathbb{E}_{\gamma_1,1}(-x) \leq \frac{1}{1+\Gamma(1+\gamma_1)^{-1}x} \tag{2.39}$$

holds over $\mathbb{R}^+$, with optimal constants. Then it follows that

$$0 < \mathbb{E}_{\gamma_1,1}(-x) < 1, \quad x > 0. \tag{2.40}$$

The Riemann-Liouville fractional integral of the Mittag-Leffler function with special parameters also yields a function of the same kind ([48, p. 78])

$$\left(I_{a+}^{\gamma}(t-a)^{\beta-1}\mathbb{E}_{\mu,\beta}\left[\lambda(t-a)^{\mu}\right]\right)(x) = (x-a)^{\gamma+\beta-1}\mathbb{E}_{\mu,\gamma+\beta}\left[\lambda(x-a)^{\mu}\right].$$

Theorem 2.74. [67, p. 35, Theorem 1.6] *Suppose that $\gamma_2 \in \mathbb{R}$, $\gamma_1 \in (0,2)$ and, $\pi\gamma_1/2 < \mu < \min\{\pi, \pi\gamma_1\}$. Then there exists a positive constant C such that*

$$|\mathbb{E}_{\gamma_1,\gamma_2}(z)| \leq \frac{C}{1+|z|},$$

for all $\mu \leq |arg(z)| \leq \pi$ and $|z| \geq 0$.

3. Pseudo-differential operators associated with the Dunkl operator

Let us have a differential operator

$$\begin{cases} A = \frac{d}{dx}, \\ D(A) = C^1(\mathbb{R}). \end{cases}$$

We can describe A as a pseudo-differential operator using classical inverse Fourier transform $\mathcal{F}^{-1}$ as following

$$\begin{aligned} A\varphi(x) &= \int_{\mathbb{R}} A \exp(ix\lambda)\mathcal{F}[\varphi](\lambda)d\lambda \\ &= \int_{\mathbb{R}} \exp(ix\lambda)i\lambda\mathcal{F}[\varphi](\lambda)d\lambda \\ &= \int_{\mathbb{R}} \exp(ix\lambda)a(x,\lambda)\mathcal{F}[\varphi](\lambda)d\lambda. \end{aligned}$$

So, we have

$$A\varphi(x) = \int_{\mathbb{R}} \exp(ix\lambda)a(x,\lambda)\mathcal{F}[\varphi](\lambda)d\lambda \tag{3.1}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$ and the function $a(x,\lambda) = i\lambda$ is called the symbol of the operator A. Then the idea of pseudo-differential operators is to consider operators of the form (3.1) where $a(x,\lambda)$ is a more general sort of function. Thus, pseudo-differential operator is defined by

$$T_a f(x) = \int_{\mathbb{R}} \exp(ix\lambda)a(x,\lambda)\mathcal{F}[f](\lambda)d\lambda, \tag{3.2}$$

for all $f \in \mathcal{S}(\mathbb{R})$, where the symbol $a(x,\lambda)$ from following class:

Definition 3.1 (**Symbol classes** $S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$)**.** Let $m \in \mathbb{R}$ and $0 \le \rho, \delta \le 1$. If $a = a(x,\lambda)$ is in $C^\infty(\mathbb{R}\times\mathbb{R})$ and

$$\left|\partial_x^k \partial_\lambda^n a(x,\lambda)\right| \le C_{n,k}(1+|\lambda|)^{m-\rho n+\delta k}$$

for all $n,k \in \mathbb{N}$ and all $x,\lambda \in \mathbb{R}$. Then we will say that $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$.

Now, we would like to give a motivation to study pseudo-differential operators associated with the Dunkl operator. If we consider the Dunkl operator

$$D_\alpha\varphi(x) = \frac{d}{dx}\varphi(x) + \left(\alpha + \frac{1}{2}\right)\frac{\varphi(x)-\varphi(-x)}{x}$$

with $D(D_\alpha) = C^1(\mathbb{R})$, then we are not able to calculate its symbol using classical inverse Fourier transform $\mathcal{F}^{-1}$, it is clear from

$$\begin{aligned} D_\alpha\varphi(x) &= \int_{\mathbb{R}} D_\alpha \exp(ix\lambda)\mathcal{F}[\varphi](\lambda)d\lambda \\ &= \int_{\mathbb{R}} \left(i\lambda\exp(ix\lambda) + \left(\alpha+\frac{1}{2}\right)\frac{\exp(ix\lambda)-\exp(-ix\lambda)}{x}\right)\mathcal{F}[\varphi](\lambda)d\lambda, \end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. On the other hand, if we use the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ instead of classical inverse Fourier transform $\mathcal{F}^{-1}$ we obtain

$$D_\alpha\varphi(x) = \int_{\mathbb{R}} D_\alpha E_\alpha(x,\lambda)\mathcal{F}_\alpha[\varphi](\lambda)d\mu_\alpha(\lambda) = \int_{\mathbb{R}} E_\alpha(x,\lambda)i\lambda\mathcal{F}_\alpha[\varphi](\lambda)d\mu_\alpha(\lambda).$$

So the Dunkl operator D_α is a pseudo-differential operator with symbol $a(x,\lambda) = i\lambda$ in the Dunkl setting.

3.1. **Pseudo-differential and amplitude operators on Schwartz spaces.** In this section, We define amplitude, adjoint and transpose operators and prove that pseudo-differential, amplitude, adjoint and transpose operators are linear transformations on the Schwartz spaces.

Lemma 3.2. *Let $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. Then*

- *for every $x \in \mathbb{R}$, the function $\lambda \longmapsto a(x,\lambda)f(\lambda)$ is belongs to the $\mathcal{S}(\mathbb{R})$. Moreover, we have*

$$\sup_{x\in\mathbb{R}} p_{n,k}(a(x,\cdot)f) < +\infty.$$

- *for every $\lambda \in \mathbb{R}$, the function $x \longmapsto a(x,\lambda)f(x)$ is belongs to the $\mathcal{S}(\mathbb{R})$, i.e.*

$$p_{n,k}(a(\cdot,\lambda)f) \le C_{n,k}(1+|\lambda|)^{m+\delta n}.$$

Bewijs. Let $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. Then for every $x \in \mathbb{R}$, the function $\lambda \longmapsto a(x,\lambda)f(\lambda)$ is belongs to the $\mathcal{S}(\mathbb{R})$. Since

$$\begin{aligned}
\left|\lambda^k \frac{d^n}{d\lambda^n}\left(a(x,\lambda)f(\lambda)\right)\right| &= \left|\lambda^k \sum_{i=0}^n C_n^i \frac{d^{n-i}}{d\lambda^{n-i}}a(x,\lambda)\frac{d^i}{d\lambda^i}f(\lambda)\right| \\
&\le \sum_{i=0}^n C_n^i \left|\lambda^k \frac{d^{n-i}}{d\lambda^{n-i}}a(x,\lambda)\frac{d^i}{d\lambda^i}f(\lambda)\right| \\
&\le \sum_{i=0}^n C_n^i |\lambda|^k (1+|\lambda|)^{m-\rho(n-i)}\left|\frac{d^i}{d\lambda^i}f(\lambda)\right| \\
&\le \sum_{i=0}^n C_n^i (1+|\lambda|)^{k+m+\rho i}\left|\frac{d^i}{d\lambda^i}f(\lambda)\right| \\
&\le \sum_{i=0}^n C_n^i \left|(1+|\lambda|)^{k+\ell i}\frac{d^i}{d\lambda^i}f(\lambda)\right|
\end{aligned}$$

and

$$p_{n,k}(a(x,\cdot)f) = \sup_{\lambda\in\mathbb{R}}\left|\lambda^k \frac{d^n}{d\lambda^n}\left(a(x,\lambda)f(\lambda)\right)\right| \le \sum_{i=0}^n C_n^i \sup_{\lambda\in\mathbb{R}}\left|(1+|\lambda|)^{k+\ell i}\frac{d^i}{d\lambda^i}f(\lambda)\right| < +\infty,$$

where for every fixed m and ρ we can find positive integer ℓ, which satisfy the inequality $m+\rho i < \ell i$. Moreover, we have

$$\sup_{x\in\mathbb{R}} p_{n,k}(a(x,\cdot)f) \le \sum_{i=0}^n C_n^i \sup_{\lambda\in\mathbb{R}}\left|(1+|\lambda|)^{k+\ell i}\frac{d^i}{d\lambda^i}f(\lambda)\right| < +\infty. \tag{3.3}$$

Now, let us have the function $x \longmapsto a(x,\lambda)f(x)$ for every $\lambda \in \mathbb{R}$. Then we are able to calculate

$$\begin{aligned}\left|x^k \frac{d^n}{dx^n}\left(a(x,\lambda)f(x)\right)\right| &= \left|x^k \sum_{i=0}^{n} C_n^i \frac{d^{n-i}}{dx^{n-i}} a(x,\lambda) \frac{d^i}{dx^i} f(x)\right| \\ &\leq \sum_{i=0}^{n} C_n^i \left|x^k \frac{d^{n-i}}{dx^{n-i}} a(x,\lambda) \frac{d^i}{dx^i} f(x)\right| \\ &\leq \sum_{i=0}^{n} C_n^i (1+|\lambda|)^{m+\delta(n-i)} \left|x^k \frac{d^i}{dx^i} f(x)\right| \\ &\leq (1+|\lambda|)^{m+\delta n} \sum_{i=0}^{n} C_n^i \left|x^k \frac{d^i}{dx^i} f(x)\right|\end{aligned}$$

and

$$\begin{aligned}p_{n,k}(a(\cdot,\lambda)f) &= \sup_{x\in\mathbb{R}} \left|x^k \frac{d^n}{dx^n}\left(a(x,\lambda)f(x)\right)\right| \\ &\leq (1+|\lambda|)^{m+\delta n} \sum_{i=0}^{n} C_n^i \sup_{x\in\mathbb{R}} \left|x^k \frac{d^i}{dx^i} f(x)\right| \\ &\leq C_{n,k}(1+|\lambda|)^{m+\delta n}.\end{aligned}$$

□

Lemma 3.3. *Let $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f_j \to f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$. Then we obtain*

- *$a(x,\cdot)f_j \to a(x,\cdot)f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$, for every $x \in \mathbb{R}$;*
- *$a(\cdot,\lambda)f_j \to a(\cdot,\lambda)f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$, for every fixed $\lambda \in \mathbb{R}$.*

Bewijs. Let us show that for every $x \in \mathbb{R}$, we obtain $a(x,\cdot)f_j \to a(x,\cdot)f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$. Using previous calculations, we obtain

$$\begin{aligned}p_{n,k}(a(x,\cdot)f_j - a(x,\cdot)f) &= \sup_{\lambda\in\mathbb{R}} \left|\lambda^k \frac{d^n}{d\lambda^n}\left(a(x,\lambda)(f_j-f)(\lambda)\right)\right| \\ &\leq \sum_{i=0}^{n} C_n^i \sup_{\lambda\in\mathbb{R}} \left|(1+|\lambda|)^{k+\ell i} \frac{d^i}{d\lambda^i}(f_j-f)(\lambda)\right| \\ &\to 0\end{aligned}$$

as $j \to \infty$ for all $x \in \mathbb{R}$. However we have $a(\cdot,\lambda)f_j \to a(\cdot,\lambda)f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$ only for every fixed $\lambda \in \mathbb{R}$, i.e.

$$\begin{aligned}p_{n,k}(a(\cdot,\lambda)f_j - a(\cdot,\lambda)f) &= \sup_{x\in\mathbb{R}} \left|x^k \frac{d^n}{dx^n}\left(a(x,\lambda)(f_j-f)(x)\right)\right| \\ &\leq (1+|\lambda|)^{m+\delta n} \sum_{i=0}^{n} C_n^i \sup_{x\in\mathbb{R}} \left|x^k \frac{d^i}{dx^i}(f_j-f)(x)\right| \\ &\to 0\end{aligned}$$

as $j \to \infty$. □

Lemma 3.4. *We assume that* $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ *and* $f \in \mathcal{S}(\mathbb{R})$*. Then*

$$\sup_{\lambda\in\mathbb{R}} \left|(1+|\lambda|)^\ell \partial_x^k D^n_{\alpha,\lambda}\left(a(x,\lambda)f(\lambda)\right)\right| < +\infty$$

for all $n, k, \ell \in \mathbb{N}$*.*

Bewijs. Let assumptions of the lemma holds. Then using Mean Value Theorem as following

$$D_{\alpha,\lambda}\left[a(x,\lambda)f(\lambda)\right] = \partial_\lambda\left[a(x,\lambda)f(\lambda)\right] + (2\alpha+1)\partial_\lambda\left[a(x,c)f(c)\right]$$

for some $c \in (-x, x)$, we are able to calculate

$$D^n_{\alpha,\lambda}\left[a(x,\lambda)f(\lambda)\right] = \partial^n_\lambda\left[a(x,\lambda)f(\lambda)\right] + (2\alpha+1)\partial^n_\lambda\left[a(x,c)f(c)\right]$$

and

$$\begin{aligned}(1+|\lambda|)^\ell \partial_x^k D^n_{\alpha,\lambda}\left[a(x,\lambda)f(\lambda)\right] = (1+|\lambda|)^\ell \partial_x^k \partial^n_\lambda\left[a(x,\lambda)f(\lambda)\right] \\ + (2\alpha+1)(1+|\lambda|)^\ell \partial_x^k \partial^n_\lambda\left[a(x,c)f(c)\right].\end{aligned}$$

Then taking absolute value from last equation and supremum respect to the variable $\lambda \in \mathbb{R}$ we obtain

$$\begin{aligned}\sup_{\lambda\in\mathbb{R}}\left|(1+|\lambda|)^\ell \partial_x^k D^n_{\alpha,\lambda}\left[a(x,\lambda)f(\lambda)\right]\right| &\le \sup_{\lambda\in\mathbb{R}}\left|(1+|\lambda|)^\ell \partial_x^k \partial^n_\lambda\left[a(x,\lambda)f(\lambda)\right]\right| \\ &+ (2\alpha+1)\sup_{\lambda\in\mathbb{R}}\left|(1+|\lambda|)^\ell \partial_x^k \partial^n_\lambda\left[a(x,c)f(c)\right]\right| \\ &\le 2(\alpha+1)\sup_{\lambda\in\mathbb{R}}\left|(1+|\lambda|)^\ell \partial_x^k \partial^n_\lambda\left[a(x,\lambda)f(\lambda)\right]\right|.\end{aligned}$$

Now, let us prove that

$$\sup_{\lambda\in\mathbb{R}}\left|(1+|\lambda|)^\ell \partial_x^k \partial^n_\lambda\left[a(x,\lambda)f(\lambda)\right]\right| < +\infty.$$

Indeed

$$\partial_x^k \partial^n_\lambda\left[a(x,\lambda)f(\lambda)\right] = \partial^n_\lambda\left[\partial_x^k a(x,\lambda)f(\lambda)\right] = \sum_{i=0}^n C_n^i \partial_\lambda^{n-i}\partial_x^k a(x,\lambda)\partial_\lambda^i f(\lambda)$$

and

$$\begin{aligned}\left|(1+|\lambda|)^\ell \partial_x^k \partial^n_\lambda\left[a(x,\lambda)f(\lambda)\right]\right| &\le \sum_{i=0}^n C_n^i\left|\partial_\lambda^{n-i}\partial_x^k a(x,\lambda)(1+|\lambda|)^\ell \partial_\lambda^i f(\lambda)\right| \\ &\le \sum_{i=0}^n C_n^i C_{n-i,k}(1+|\lambda|)^{m-\rho(n-i)+\delta k}\left|(1+|\lambda|)^\ell \partial_\lambda^i f(\lambda)\right| \\ &= \sum_{i=0}^n C_n^i C_{n-i,k}(1+|\lambda|)^{m+\ell-\rho(n-i)+\delta k}\left|\partial_\lambda^i f(\lambda)\right| \\ &< +\infty,\end{aligned}$$

since $f \in \mathcal{S}(\mathbb{R})$. □

Lemma 3.5. *We suppose that $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f_j \to f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$. Then*

$$\sup_{\lambda\in\mathbb{R}} \left|(1+|\lambda|)^\ell \partial_x^k D^n_{\alpha,\lambda}\left(a(x,\lambda)(f_j - f)(\lambda)\right)\right| \to 0$$

as $j \to \infty$ for all $n, k, \ell \in \mathbb{N}$.

Bewijs. The proof of this lemma is the same as the proof of Lemma 3.4. □

If $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$, it is convenient to denote by T_a the corresponding pseudo-differential operator defined by

$$T_a f(x) := \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda)\mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda), \tag{3.4}$$

where $\mathcal{F}_\alpha$ is the Dunkl transform (2.23) of f, E_α is the Dunkl kernel (2.12), and $d\mu_\alpha$ a weighted Lebesgue measure (2.21) on $\mathbb{R}$.

Theorem 3.6 (Pseudo-differential operators on $\mathcal{S}(\mathbb{R})$). *Assume that $f \in \mathcal{S}(\mathbb{R})$ and $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$. Then $T_a f \in \mathcal{S}(\mathbb{R})$.*

Bewijs. Let $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. To show absolute convergence of the integral (3.4), we calculate

$$\begin{aligned} |T_a f(x)| &\le \int_{\mathbb{R}} |a(x,\lambda)| \cdot |\mathcal{F}_\alpha[f](\lambda)|\, d\mu_\alpha(\lambda) \\ &\le C_k \int_{\mathbb{R}} \frac{|a(x,\lambda)|}{(1+|\lambda|)^k} d\mu_\alpha(\lambda) \\ &\le C_k \cdot C \int_{\mathbb{R}} \frac{(1+|\lambda|)^m}{(1+|\lambda|)^k} d\mu_\alpha(\lambda) \\ &< +\infty, \end{aligned}$$

as m and α fixed here, we can find suitable $k \in \mathbb{N}$. Using Lebesgue's dominated convergence theorem and properties of the Dunkl kernel E_α we obtain

$$\frac{d}{dx} T_a f(x) = \int_{\mathbb{R}} \frac{d}{dx}\left(E_\alpha(x,\lambda) a(x,\lambda)\right) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$

$$= \int_{\mathbb{R}} \frac{d}{dx} E_\alpha(x,\lambda) a(x,\lambda)\mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) + \int_{\mathbb{R}} E_\alpha(x,\lambda) \frac{d}{dx} a(x,\lambda)\mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)$$

and

$$\left|\frac{d}{dx} T_a f(x)\right| \le \int_{\mathbb{R}} |a(x,\lambda)| \cdot |\lambda \mathcal{F}_\alpha[f](\lambda)| d\mu_\alpha(\lambda) + \int_{\mathbb{R}} \left|\frac{d}{dx} a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right| d\mu_\alpha(\lambda).$$

Thus $\left|\frac{d}{dx} T_a f(x)\right| < +\infty$. The same is true for all of its other derivatives, it is obvious from last equation, which implies that $T_a f \in C^\infty(\mathbb{R})$. Let us show now that $T_a f \in \mathcal{S}_\alpha(\mathbb{R})$. In fact we have

$$\begin{aligned} (-ix)^k T_a f(x) &= (-1)^k \int_{\mathbb{R}} (ix)^k E_\alpha(x,\lambda) a(x,\lambda)\mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \\ &= (-1)^k \int_{\mathbb{R}} D^k_{\alpha,\lambda} E_\alpha(x,\lambda) a(x,\lambda)\mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \end{aligned}$$

$$= \int_{\mathbb{R}} E_\alpha(x,\lambda) D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right) d\mu_\alpha(\lambda).$$

Then for for arbitrary $k, n \in \mathbb{N}$ we obtain

$$\begin{aligned}
\frac{d^n}{dx^n}\left(x^k T_a f(x)\right) &= \frac{1}{(-i)^k}\int_{\mathbb{R}} \partial_x^n \left[E_\alpha(x,\lambda) D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right)\right] d\mu_\alpha(\lambda) \\
&= \frac{1}{(-i)^k}\int_{\mathbb{R}} \sum_{i=0}^n C_n^i \partial_x^{n-i} E_\alpha(x,\lambda)\partial_x^i D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right) d\mu_\alpha(\lambda) \\
&= \sum_{i=0}^n \frac{C_n^i}{(-i)^k}\int_{\mathbb{R}} \partial_x^{n-i} E_\alpha(x,\lambda)\partial_x^i D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right) d\mu_\alpha(\lambda)
\end{aligned}$$

which implies

$$\begin{aligned}
\left|\frac{d^n}{dx^n}\left(x^k T_a f(x)\right)\right| &\leq \sum_{i=0}^n C_n^i \int_{\mathbb{R}} \left|\partial_x^{n-i} E_\alpha(x,\lambda)\partial_x^i D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right)\right| d\mu_\alpha(\lambda) \\
&\leq \sum_{i=0}^n C_n^i \int_{\mathbb{R}} \left|\lambda^{n-i}\partial_x^i D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right)\right| d\mu_\alpha(\lambda) \\
&\leq \sum_{i=0}^n \frac{C_n^i C}{2^{\alpha+1}\Gamma(\alpha+1)} \sup_{\lambda\in\mathbb{R}} \left|(1+|\lambda|)^{n-i+\ell}\partial_x^i D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\right)\right| \\
&< +\infty
\end{aligned}$$

for all $n, k \in \mathbb{N}$, where

$$C = \int_{\mathbb{R}} \frac{|x|^{2\alpha+1}}{(1+|x|)^\ell} dx < +\infty, \quad \ell > 2(\alpha+1) \quad \text{and} \quad \ell \in \mathbb{N}. \tag{3.5}$$

There we have used Lemma 3.4. Then taking into account inequality (3.3), we can see that $\frac{d^n}{dx^n}\left(x^k T_a f(x)\right)$ is bounded for all $k, n \in \mathbb{N}$. This completes the proof. □

Theorem 3.6 shows that the operator T_a is a linear operator defined on the Schwartz space $\mathcal{S}(\mathbb{R})$. Actually, can prove that T_a is a linear continuous operator on the Schwartz space $\mathcal{S}(\mathbb{R})$.

Proposition 3.7. *Suppose that $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator T_a is a continuous linear operator on $\mathcal{S}(\mathbb{R})$.*

Bewijs. Let $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f_j \to f$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$. Then using Proposition 2.49 and Lemma 3.5 we have

$$\begin{aligned}
\left|\frac{d^n}{dx^n}\left(x^k (T_a f_j - T_a f)(x)\right)\right| &= \left|\frac{d^n}{dx^n}\left(x^k T_a (f_j - f)(x)\right)\right| \\
&\leq \sum_{i=0}^n C_n^i \int_{\mathbb{R}} \left|\lambda^{n-i}\partial_x^i D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f_j - f](\lambda)\right)\right| d\mu_\alpha(\lambda) \\
&\leq \sum_{i=0}^n \frac{C_n^i C}{2^{\alpha+1}\Gamma(\alpha+1)} \sup_{\lambda\in\mathbb{R}} \left|(1+|\lambda|)^{n-i+\ell}\partial_x^i D^k_{\alpha,\lambda}\left(a(x,\lambda)\mathcal{F}_\alpha[f_j - f](\lambda)\right)\right| \to 0,
\end{aligned}$$

for every fixed $x \in \mathbb{R}$, where C is constant defined by (3.5). □

Proposition 3.8. *Assume that we have a sequence of symbols $a_i \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ which satisfies*

$$\left|\partial_x^k \partial_\lambda^n a_i(x,\lambda)\right| \le C_{n,k}(1+|\lambda|)^{m-\rho n+\delta k}$$

for all k, n, all $x,\lambda \in \mathbb{R}$, and all i, with constants $C_{n,k}$ independent from x,λ and i, and $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ such that $a_i(x,\lambda)$ and all of its derivatives converge to $a(x,\lambda)$ and its derivatives, respectively, pointwise as $i\to\infty$. Then for any $f \in \mathcal{S}(\mathbb{R})$ we have

$$T_{a_i} f \to T_a f \quad \textit{as} \quad i\to\infty$$

in $\mathcal{S}(\mathbb{R})$.

Bewijs. From Theorem 3.6, for every $a, a_i \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$ we have $T_a f, T_{a_i} f \in \mathcal{S}(\mathbb{R})$. Then using Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}\lim_{i\to\infty} T_{a_i} f(x) &= \lim_{i\to\infty} \int_{\mathbb{R}} E_\alpha(x,\lambda) a_i(x,\lambda)\mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda)\mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda)\\ &= T_a f(x)\end{aligned}$$

since there is an integrable function $(1+|\lambda|)^m \mathcal{F}_\alpha[f](\lambda) \in L^1(\mathbb{R})$ such that

$$|E_\alpha(x,\lambda) a_i(x,\lambda)\mathcal{F}_\alpha[f](\lambda)| \le C(1+|\lambda|)^m \left|\mathcal{F}_\alpha[f](\lambda)\right|.$$

A short calculation gives us

$$\begin{aligned}&\frac{d^n}{dx^n}\left(x^k (T_{a_i} f - T_a f)(x)\right)\\ &\qquad = \sum_{i=0}^n C_n^i i^k \int_{\mathbb{R}} \partial_x^{n-i} E_\alpha(x,\lambda)\partial_x^i D^k_{\alpha,\lambda}((a_i(x,\lambda)-a(x,\lambda))\mathcal{F}_\alpha[f](\lambda)) d\mu_\alpha(\lambda).\end{aligned}$$

Then using Lemma 3.5 we have

$$\begin{aligned}&\left|\frac{d^n}{dx^n}\left(x^k (T_{a_i} f - T_a f)(x)\right)\right|\\ &\le \sum_{i=0}^n C_n^i \int_{\mathbb{R}} \left|\partial_x^{n-i} E_\alpha(x,\lambda)\partial_x^i D^k_{\alpha,\lambda}((a_i(x,\lambda)-a(x,\lambda))\mathcal{F}_\alpha[f](\lambda))\right| d\mu_\alpha(\lambda)\\ &\le \sum_{i=0}^n C_n^i \int_{\mathbb{R}} |\lambda|^{n-i} \left|\partial_x^i D^k_{\alpha,\lambda}((a_i(x,\lambda)-a(x,\lambda))\mathcal{F}_\alpha[f](\lambda))\right| d\mu_\alpha(\lambda)\\ &\le \sum_{i=0}^n \frac{C_n^i C}{2^{\alpha+1}\Gamma(\alpha+1)} \sup_{\lambda\in\mathbb{R}} \left|(1+|\lambda|)^{n-i+\ell}\partial_x^i D^k_{\alpha,\lambda}\left((a_i(x,\lambda)-a(x,\lambda))\mathcal{F}_\alpha[f](\lambda)\right)\right|\\ &\to 0, \quad \text{as} \quad i\to\infty.\end{aligned}$$

for every fixed $x \in \mathbb{R}$, where C is constant defined by (3.5). □

Let us introduce more general symbol class that introduced in Definition 3.1.

Definition 3.9. Let $m \in \mathbb{R}$ and $0 \le \rho, \delta_1, \delta_2 \le 1$. The class $S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ is the space of all functions $a = a(x, y, \lambda)$ which are $a \in C^\infty(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ and satisfy

$$|\partial^n_\lambda \partial^\ell_y \partial^k_x a(x, y, \lambda)| \le C_{n,k,\ell}(1 + |\lambda|)^{m-\rho n+\delta_1 k+\delta_2 \ell}$$

for all $x, y, \lambda \in \mathbb{R}$ and all $n, k, \ell \in \mathbb{N}$.

By analogy of (3.4), for all $f \in \mathcal{S}(\mathbb{R})$ we may also formally define corresponding operator

$$Af(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x, \lambda)E_\alpha(-y, \lambda)a(x, y, \lambda)f(y)d\mu_\alpha(y)d\mu_\alpha(\lambda). \tag{3.6}$$

to the $a \in S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, which is called *amplitude.*

Theorem 3.10. *Let $a \in S^m_{\rho,\delta_1,\delta_2}(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$ and $\delta_2 < 1$. Then the operator A is a linear continuous operator on $\mathcal{S}(\mathbb{R})$.*

Bewijs. First we prove that the function

$$h_a(x, \lambda) = \int_{\mathbb{R}} E_\alpha(-y, \lambda)a(x, y, \lambda)f(y)d\mu_\alpha(y)$$

is rapidly decreasing in both variable $x, \lambda \in \mathbb{R}$ and $h_a \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$. To see it we calculate

$$\begin{aligned}(-i\lambda)^N \int_{\mathbb{R}} E_\alpha(-y, \lambda)a(x, y, \lambda)f(y)d\mu_\alpha(y) &= \int_{\mathbb{R}}(-i\lambda)^N E_\alpha(-y, \lambda)a(x, y, \lambda)f(y)d\mu_\alpha(y)\\ &= \int_{\mathbb{R}} D^N_{\alpha,y}E_\alpha(-y, \lambda)a(x, y, \lambda)f(y)d\mu_\alpha(y)\\ &= \int_{\mathbb{R}} E_\alpha(-y, \lambda)D^N_{\alpha,y}(a(x, y, \lambda)f(y))d\mu_\alpha(y)\end{aligned}$$

and

$$\begin{aligned}&|\lambda|^N \left|\int_{\mathbb{R}} E_\alpha(-y, \lambda)a(x, y, \lambda)f(y)d\mu_\alpha(y)\right|\\ &\le \int_{\mathbb{R}} |D^N_{\alpha,y}(a(x, y, \lambda)f(y))|d\mu_\alpha(y)\\ &= \int_{\mathbb{R}} \frac{1}{(1 + |y|)^j}|(1 + |y|)^j D^N_{\alpha,y}(a(x, y, \lambda)f(y))|d\mu_\alpha(y)\\ &\le \sup_{y\in\mathbb{R}} |(1 + |y|)^j D^N_{\alpha,y}(a(x, y, \lambda)f(y))| \int_{\mathbb{R}} \frac{1}{(1 + |y|)^j}d\mu_\alpha(y)\\ &\le C_\alpha \sup_{y\in\mathbb{R}} \left|(1 + |y|)^j \frac{d^N}{dy^N}(a(x, y, \lambda)f(y))\right|\\ &\le C_{\alpha,N,j}(1 + |\lambda|)^{m+\delta_2 N}.\end{aligned}$$

Thus,

$$|h_a(x, \lambda)| \le C_{\alpha,N,j}(1 + |\lambda|)^{m-(1-\delta_2)N}$$

and $h_a(x, \lambda)$ is rapidly decreasing if $\delta_2 < 1$. Similarly we have

$$|\partial^n_x h_a(x, \lambda)| \le C_{\alpha,N,j,n}(1 + |\lambda|)^{m+\delta_1 n-(1-\delta_2)N}$$

for all $n \in \mathbb{N}$, so $h_a(\cdot, \lambda)$ belongs to $\mathcal{S}(\mathbb{R})$. Using same technique with little modifications we obtain

$$\begin{aligned}
\lambda^k h_a(x,\lambda) &= i^k \int_{\mathbb{R}} (-i\lambda)^k E_\alpha(-y,\lambda) a(x,y,\lambda) f(y) d\mu_\alpha(y) \\
&= i^k \int_{\mathbb{R}} D^k_{\alpha,y} E_\alpha(-y,\lambda) a(x,y,\lambda) f(y) d\mu_\alpha(y) \\
&= i^k \int_{\mathbb{R}} E_\alpha(-y,\lambda) D^k_{\alpha,y}(a(x,y,\lambda) f(y)) d\mu_\alpha(y)
\end{aligned}$$

and

$$\partial^n_\lambda(\lambda^k h_a(x,\lambda)) = i^k \int_{\mathbb{R}} \partial^n_\lambda \left[E_\alpha(-y,\lambda) D^k_{\alpha,y}(a(x,y,\lambda) f(y))\right] d\mu_\alpha(y).$$

Hence, Lemma 3.4 leads that $\partial^n_\lambda(\lambda^k h_a(x,\lambda))$ is bounded for arbitrary $n, k \in \mathbb{N}$ and $h_a \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$. So,

$$\begin{aligned}
Af(x) &= \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\lambda) E_\alpha(-y,\lambda) a(x,y,\lambda) f(y) d\mu_\alpha(y) d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda) h_a(x,\lambda) d\mu_\alpha(\lambda)
\end{aligned}$$

is absolutely integrable for $f \in \mathcal{S}(\mathbb{R})$, if $\delta_2 < 1$ and

$$\frac{d^n}{dx^n}(x^k Af)(x) = \frac{1}{i^k} \sum_{i=0}^{n} C^i_n \int_{\mathbb{R}} \partial^{n-i}_x E_\alpha(x,\lambda) \partial^i_x D^k_{\alpha,\lambda} h_a(x,\lambda) d\mu_\alpha(\lambda)$$

is bounded. Thus, $Af \in \mathcal{S}(\mathbb{R})$.

After successfully proving the first part of the theorem, we now turn our attention to the second part, which deals with continuity of the operator A on Schwartz spaces. Assume that there exists a sequence $\{f_j\}_{j=1}^{\infty}$, which converges to f in $\mathcal{S}(\mathbb{R})$. Then we need to show that $Af_j \to Af$ in $\mathcal{S}(\mathbb{R})$ as $j \to \infty$. Let's first prove that if $f_j \to f$, as $j \to \infty$, then $h^j_a \to h_a$ in $\mathcal{S}(\mathbb{R})$ fro fixed $x \in \mathbb{R}$, indeed from

$$\partial^n_\lambda(\lambda^k(h^j_a - h_a)(x,\lambda)) = i^k \int_{\mathbb{R}} \partial^n_\lambda E_\alpha(-y,\lambda) D^k_{\alpha,y}(a(x,y,\lambda)(f_j - f)(y)) d\mu_\alpha(y)$$

we have

$$\begin{aligned}
\left|\partial^n_\lambda(\lambda^k(h^j_a - h_a)(x,\lambda))\right| &\leq \int_{\mathbb{R}} \left|\partial^n_\lambda E_\alpha(-y,\lambda) D^k_{\alpha,y}(a(x,y,\lambda)(f_j - f)(y))\right| d\mu_\alpha(y) \\
&\leq \int_{\mathbb{R}} |y|^n \left|D^k_{\alpha,y}(a(x,y,\lambda)(f_j - f)(y))\right| d\mu_\alpha(y) \\
&\leq C_\alpha \sup_{y \in \mathbb{R}} \left|(1+|y|)^\ell D^k_{\alpha,y}(a(x,y,\lambda)(f_j - f)(y))\right| \\
&\leq C_\alpha \sup_{y \in \mathbb{R}} \left|(1+|y|)^\ell \partial^k_y(a(x,y,\lambda)(f_j - f)(y))\right| \\
&\leq \sum_{i=0}^{k} C_\alpha (1+|\lambda|)^i p_{k-i,\ell}(f_j - f)(y),
\end{aligned}$$

which affirms our statement. Then from

$$\frac{d^n}{dx^n}(x^k A(f_j - f))(x) = \frac{1}{i^k}\sum_{i=0}^{n} C_n^i \int_{\mathbb{R}} \partial_x^{n-i} E_\alpha(x,\lambda)\partial_x^i D_{\alpha,\lambda}^k \left(h_a^j(x,\lambda) - h_a(x,\lambda)\right) d\mu_\alpha(\lambda)$$

taking absolute value we obtain

$$\begin{aligned}
&\left|\frac{d^n}{dx^n}(x^k A(f_j - f))(x)\right| \\
&\leq \sum_{i=0}^{n} C_n^i \int_{\mathbb{R}} \left|\partial_x^{n-i} E_\alpha(x,\lambda)\partial_x^i D_{\alpha,\lambda}^k \left(h_a^j(x,\lambda) - h_a(x,\lambda)\right)\right| d\mu_\alpha(\lambda) \\
&\leq \sum_{i=0}^{n} C_n^i \int_{\mathbb{R}} |\lambda|^{n-i} \left|\partial_x^i D_{\alpha,\lambda}^k \left(h_a^j(x,\lambda) - h_a(x,\lambda)\right)\right| d\mu_\alpha(\lambda) \\
&\leq \sum_{i=0}^{n} C_{\alpha,n}^i \sup_{\lambda\in\mathbb{R}} \left|(1+|\lambda|)^\ell \partial_x^i D_{\alpha,\lambda}^k \left(h_a^j(x,\lambda) - h_a(x,\lambda)\right)\right|.
\end{aligned}$$

This proves our statement. □

We are now in a good position to compute adjoints of pseudo-differential operators. We say that T_p^* is the **adjoint** ([35, Chapter 8, p. 290]) of T_p if

$$\langle T_p f, g\rangle = \langle f, T_p^* g\rangle,$$

for all $f, g \in \mathcal{S}(\mathbb{R})$, where

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} d\mu_\alpha(x).$$

Lemma 3.11. *Let $p \in S_{\rho,\delta}^m(\mathbb{R}\times\mathbb{R})$. Then $T_p^* = A$, where $a(x,y,\lambda) = \overline{p(y,\lambda)}$.*

Bewijs. Assume that $f, g \in \mathcal{S}(\mathbb{R})$. Since

$$\begin{aligned}
\langle T_p f, g\rangle_{L^2(\mathbb{R},d\mu_\alpha)} &= \int_{\mathbb{R}} T_p f(x)\overline{g(x)} d\mu_\alpha(x) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)p(x,\lambda)f(y)\overline{g(x)} d\mu_\alpha(y)d\mu_\alpha(\lambda)d\mu_\alpha(x),
\end{aligned}$$

to prove this, all we need to do is reverse the order of integration. However, it's important to note that this is not a common application of Fubini's theorem, as the triple integral is typically not absolutely convergent. Instead, we utilize Fubini's theorem on the double integral

$$\langle T_p f, g\rangle_{L^2(\mathbb{R},d\mu_\alpha)} = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)p(x,\lambda)\mathcal{F}_\alpha[f](\lambda)\overline{g(x)} d\mu_\alpha(\lambda)d\mu_\alpha(x)$$

that is absolutely convergent, to obtain

$$\langle T_p f, g\rangle_{L^2(\mathbb{R},d\mu_\alpha)} = \int_{\mathbb{R}} h_p(\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda),$$

where

$$h_p(\lambda) = \int_{\mathbb{R}} E_\alpha(x,\lambda)p(x,\lambda)\overline{g(x)} d\mu_\alpha(x).$$

The function h_p is a rapidly decreasing function by proof of Theorem 3.10, so we can apply Multiplication formula for the Dunkl transform (Lemma 2.50) as following

$$\begin{aligned}\langle T_p f, g\rangle_{L^2(\mathbb{R},d\mu_\alpha)} &= \int_{\mathbb{R}} h_p(\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}} \mathcal{F}_\alpha[h_p](\lambda)f(\lambda)d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}} \mathcal{F}_\alpha[h_p](y)f(y)d\mu_\alpha(y)\\ &= \langle f, T_p^* g\rangle_{L^2(\mathbb{R},d\mu_\alpha)}.\end{aligned}$$

Therefore

$$\begin{aligned}\overline{T_p^* g(y)} = \mathcal{F}_\alpha[h_p](y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)p(x,\lambda)\overline{g(x)}d\mu_\alpha(x)d\mu_\alpha(\lambda)\\ &= \overline{\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(y,\lambda)E_\alpha(-x,\lambda)\overline{p(x,\lambda)}g(x)d\mu_\alpha(x)d\mu_\alpha(\lambda)}\end{aligned}$$

and

$$T_p^* g(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)\overline{p(y,\lambda)}g(y)d\mu_\alpha(y)d\mu_\alpha(\lambda),$$

so that $T_p^* = A$ with $a(x,y,\lambda) = \overline{p(y,\lambda)}$ as claimed. □

Corollary 3.12. *Let $p \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $\delta < 1$. Then T_p^* is a linear continuous map on $\mathcal{S}(\mathbb{R})$.*

We say that T_p' is the **transpose** ([35, Chapter 8, p. 289]) of T_p if

$$\langle T_p f, g\rangle = \langle f, T_p' g\rangle,$$

for all $f, g \in \mathcal{S}(\mathbb{R})$, where

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)d\mu_\alpha(x).$$

Lemma 3.13. *Let $p \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$. Then $T_p' = A$, where $a(x,y,\lambda) = p(y,-\lambda)$.*

Bewijs. The proof of this lemma is same as for Lemma 3.11, except last part. Suppose that $f, g \in \mathcal{S}(\mathbb{R})$. Then following proof of Lemma 3.11, we obtain

$$\begin{aligned}T_p' g(y) = \mathcal{F}_\alpha[h_p](y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)p(x,\lambda)g(x)d\mu_\alpha(x)d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(y,\lambda)E_\alpha(-x,\lambda)p(x,-\lambda)g(x)d\mu_\alpha(x)d\mu_\alpha(\lambda).\end{aligned}$$

Hence,

$$T_a' g(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)p(y,-\lambda)g(y)d\mu_\alpha(y)d\mu_\alpha(\lambda),$$

so that $T_p' = A$ with $a(x,y,\lambda) = p(y,-\lambda)$. □

Corollary 3.14. *Let $p \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $\delta < 1$. Then T_p' is a linear continuous map on $\mathcal{S}(\mathbb{R})$.*

Before, we proved that the pseudo-differential operator T_a with symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is a continuous linear operator on $\mathcal{S}(\mathbb{R})$ (see Proposition 3.7) in the sense that if $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R})$ then $T_a\varphi_j \to T_a\varphi$ in $\mathcal{S}(\mathbb{R})$. Suppose there is an adjoint T_a^* of the operator T_a. Then we can extend T_a to act on distributions as following:

Definition 3.15. Let $u \in \mathcal{S}'(\mathbb{R})$. Then for all $\varphi \in \mathcal{S}(\mathbb{R})$ we define $T_a u$ by the formula

$$(T_a u)(\varphi) := u(\overline{T_a^*\overline{\varphi}}),$$

where

$$\overline{T_a^*\overline{\varphi}}(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)\varphi(x)d\mu_\alpha(x)d\mu_\alpha(\lambda).$$

The linear functional $T_a u$ on $\mathcal{S}'(\mathbb{R})$ defined in this way is continuous on $\mathcal{S}'(\mathbb{R})$ since T_a^* is continuous.

Proposition 3.16. *Let $m \in \mathbb{R}$, $0 \le \rho \le 1$, and $0 \le \delta < 1$. If $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ and $u \in \mathcal{S}'(\mathbb{R})$ then $T_a u \in \mathcal{S}'(\mathbb{R})$. Moreover, the operator $T_a : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is continuous.*

Bewijs. Let $u \in \mathcal{S}'(\mathbb{R})$. Let us prove linearity of the functional $T_a u$. Using definition of the tempered distributions, we have

$$\begin{aligned}(T_a u)(\alpha\varphi + \beta\psi) &= u(\overline{T_a^*(\overline{\alpha\varphi} + \overline{\beta\psi})}) = u(\alpha\overline{T_a^*\overline{\varphi}} + \beta\overline{T_a^*\overline{\psi}}) \\ &= \alpha u(\overline{T_a^*\overline{\varphi}}) + \beta u(\overline{T_a^*\overline{\psi}}) = \alpha(T_a u)(\varphi) + \beta(T_a u)(\psi)\end{aligned}$$

for all $\alpha, \beta \in \mathbb{C}$ and all $\varphi, \psi \in \mathcal{S}(\mathbb{R})$. Now we prove continuity of the functional $T_a u$. Let we have $\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R})$, then we have $\overline{T_a^*\overline{\varphi_j}} \to \overline{T_a^*\overline{\varphi}}$ in $\mathcal{S}(\mathbb{R})$ and

$$\lim_{j\to\infty}(T_a u)(\varphi_j) = \lim_{j\to\infty} u(\overline{T_a^*\overline{\varphi_j}}) = u(\lim_{j\to\infty}\overline{T_a^*\overline{\varphi_j}}) = u(\overline{T_a^*\overline{\varphi}}) = (T_a u)(\varphi).$$

These two propositions give us $T_a u \in \mathcal{S}'(\mathbb{R})$. Let we have $u_k \to u$ in $\mathcal{S}'(\mathbb{R})$. Then to show the continuity of the operator T_a it is enough to calculate

$$(T_a u_k)(\varphi) = u_k(\overline{T_a^*\overline{\varphi}}) \to u(\overline{T_a^*\overline{\varphi}}) = (T_a u)(\varphi).$$

□

3.2. **Kernel of pseudo-differential operators.** Let $f \in \mathcal{S}(\mathbb{R})$. Assume that T is an integral operator on some space of functions on $\mathbb{R}$ that

$$Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)d\mu_\alpha(y).$$

If $g \in \mathcal{S}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} Tf(x)g(x)d\mu_\alpha(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)g(x)f(y)d\mu_\alpha(y)d\mu_\alpha(x),$$

or in the language of distributions,

$$\langle Tf, g\rangle = \langle K, fg\rangle \tag{3.7}$$

If we suppose that T is a continuous linear map form $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$, then the Schwartz kernel theorem implies that there exists unique kernel $K \in \mathcal{S}'(\mathbb{R}\times\mathbb{R})$ such that (3.7) holds for all $f, g \in \mathcal{S}(\mathbb{R})$ and K is called the **distributional kernel** of T.

Now, it is easy compute the distributional kernel of a pseudo-differential operator. Indeed, if $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$, then

$$\begin{aligned}\langle T_a f, g\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)g(x)d\mu_\alpha(\lambda)d\mu_\alpha(x)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)f(y)g(x)d\mu_\alpha(y)d\mu_\alpha(\lambda)d\mu_\alpha(x)\end{aligned}$$

from which it follows that the kernel K of T_a is

$$K(x,y) = \int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)d\mu_\alpha(\lambda),$$

with the appropriate interpretation as a distributional integral. On the other hand, if we regularize the integral

$$T_a f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)f(y)d\mu_\alpha(y)d\mu_\alpha(\lambda) \tag{3.8}$$

for $f \in \mathcal{S}(\mathbb{R})$, we able to consider the kernel K as a function. Let k be a positive even integer, then we have

$$E_\alpha(-y,\lambda) = (1+\lambda^2)^{-k}(1-D^2_{\alpha,y})^k E_\alpha(-y,\lambda).$$

Then inserting this expression into (3.8), replacing $E_\alpha(-y,\lambda)$, and integrating by parts, we obtain

$$\begin{aligned}T_a f(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)f(y)d\mu_\alpha(y)d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)(1+\lambda^2)^{-k}(1-D^2_{\alpha,y})^k E_\alpha(-y,\lambda)a(x,\lambda)f(y)d\mu_\alpha(y)d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)(1+\lambda^2)^{-k}E_\alpha(-y,\lambda)a(x,\lambda)(1-D^2_{\alpha,y})^k f(y)d\mu_\alpha(y)d\mu_\alpha(\lambda)\end{aligned}$$

where $f \in \mathcal{S}(\mathbb{R})$. Then this integral is absolutely convergent, if we set $k > m+2(\alpha+1)$, and

$$K(x,y) = \int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)(1+\lambda^2)^{-k}a(x,\lambda)d\mu_\alpha(\lambda) \tag{3.9}$$

is a convergent integral for $k > m+2(\alpha+1)$.

Theorem 3.17 (**Kernel of a pseudo-differential operators**). *Let $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$. Then $K(x,y)$, given by (3.9), is C^∞ on $\{(x,y)\in\mathbb{R}^2 : |x| \neq |y|\}$, and*

$$|K(x,y)| \leq \frac{C_{N,\alpha}}{||x|-|y||^N}$$

for all $N \in \mathbb{N}$ and $|x| \neq |y|$.

To prove the Theorem we shall use the following simple assertion.

Lemma 3.18. *Let $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$. Then have*

$$|D^n_{\alpha,\lambda}\partial^k_x a(x,\lambda)| \leq C_{n,k,\alpha}(1+|\lambda|)^{m-\rho n+\delta k}$$

and

$$|\partial^k_\lambda D^n_{\alpha,x} a(x,\lambda)| \leq C_{n,k,\alpha}(1+|\lambda|)^{m-\rho n+\delta k}$$

for all $n, k \in \mathbb{N}$ *and all* $x, \lambda \in \mathbb{R}$.

Bewijs. Let $a \in S^m_{\rho,\delta}(\mathbb{R} \times \mathbb{R})$. Then from (2.9) we can readily see that

$$D^n_{\alpha,\lambda}\partial^k_x a(x,\lambda) = \partial^n_\lambda \partial^k_x a(x,\lambda) + (2\alpha+1)\partial^n_\lambda \partial^k_x a(x,c)$$

for some $-x < c < x$. Thus,

$$\left|D^n_\alpha \partial^k_x a(x,\lambda)\right| \leq \left|\partial^n_\lambda \partial^k_x a(x,\lambda)\right| + (2\alpha+1)\left|\partial^n_\lambda \partial^k_x a(x,c)\right| \leq C_{n,k,\alpha}(1+|\lambda|)^{m-\rho n+\delta k}$$

for all $n, k \in \mathbb{N}$ and all $x, \lambda \in \mathbb{R}$. Second inequality can be proved using same method. $\square$

Now, let us give the proof of Theorem 3.17.

Bewijs. From the kernel expression (3.9), for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
|\partial^n_y K(x,y)| &\leq \int_{\mathbb{R}} |E_\alpha(x,\lambda)\partial_y E_\alpha(-y,\lambda)(1+\lambda^2)^{-k}a(x,\lambda)|d\mu_\alpha(\lambda) \\
&\leq \int_{\mathbb{R}} |\lambda|^n(1+\lambda^2)^{-k}|a(x,\lambda)|d\mu_\alpha(\lambda) \\
&< +\infty
\end{aligned}$$

and

$$\begin{aligned}
|\partial^n_x K(x,y)| &\leq \int_{\mathbb{R}} |E_\alpha(-y,\lambda)(1+\lambda^2)^{-k}\partial_x(E_\alpha(x,\lambda)a(x,\lambda))|d\mu_\alpha(\lambda) \\
&\leq \sum_{j=0}^{n} C^j_n \int_{\mathbb{R}} |\lambda|^{n-j}(1+\lambda^2)^{-k}|\partial^j_x a(x,\lambda)|d\mu_\alpha(\lambda) \\
&< +\infty
\end{aligned}$$

for large enough $k \in \mathbb{N}$. So, $K \in C^\infty(\mathbb{R} \times \mathbb{R} \setminus \{x = y\})$. Applying Theorem 2.54 for $E_\alpha(x,\lambda)E_\alpha(-y,\lambda)$ we obtain

$$\begin{aligned}
K(x,y) &= \int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)(1+\lambda^2)^{-k}a(x,\lambda)d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(z,\lambda)(1+\lambda^2)^{-k}a(x,\lambda)d\nu_{x,-y}(z)d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{(iz)^N}(iz)^N E_\alpha(z,\lambda)(1+\lambda^2)^{-k}a(x,\lambda)d\nu_{x,-y}(z)d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{(iz)^N} D^N_{\alpha,\lambda}E_\alpha(z,\lambda)(1+\lambda^2)^{-k}a(x,\lambda)d\nu_{x,-y}(z)d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{(iz)^N} E_\alpha(z,\lambda)D^N_{\alpha,\lambda}((1+\lambda^2)^{-k}a(x,\lambda))d\nu_{x,-y}(z)d\mu_\alpha(\lambda).
\end{aligned}$$

Then

$$\begin{aligned}
|K(x,y)| &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{|z|^N}|D^N_{\alpha,\lambda}((1+\lambda^2)^{-k}a(x,\lambda))|d|\nu_{x,-y}|(z)d\mu_\alpha(\lambda) \\
&\leq \frac{C}{||x|-|y||^N}\int_{\mathbb{R}} |D^N_{\alpha,\lambda}((1+\lambda^2)^{-k}a(x,\lambda))|d\mu_\alpha(\lambda),
\end{aligned}$$

since from Theorem 2.54 it is known that $\mathrm{supp}\nu_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$ for $x, y \neq 0$, after taking into account that $\lambda \mapsto (1+\lambda^2)^{-k}$ is the even function we obtain

$$\begin{aligned} |K(x,y)| &\leq \frac{C}{||x|-|y||^N} \sum_{j=0}^{N} C_N^j \int_{\mathbb{R}} |\partial_\lambda^j (1+\lambda^2)^{-k} D_{\alpha,\lambda}^{N-j} a(x,\lambda)| d\mu_\alpha(\lambda) \\ &\leq \frac{C_{N,\alpha}}{||x|-|y||^N} \end{aligned}$$

for large enough $k \in \mathbb{N}$. □

The **singular support** of a distribution $f \in \mathcal{S}'(\mathbb{R})$ is the complement of the largest open set on which f is a C^∞ function.

Corollary 3.19 (**Singular supports**). *Let T_a is a pseudo-differential operator with symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$. Then for every $f \in \mathcal{S}'(\mathbb{R})$ we have*

$$sing\, supp\, T_a f \subset sing\, supp\, f.$$

Lemma 3.20 (**Schur's lemma**)**.** *Let $K : \mathbb{R}\times\mathbb{R} \to \mathbb{K}$ be a continuous function satisfying*

$$C_1 := \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |K(x,y)| d\mu_\alpha(y) < +\infty \quad and \quad C_2 := \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |K(x,y)| d\mu_\alpha(x) < +\infty,$$

and let A be the linear operator with Schwartz kernel K:

$$Af(x) = \int_{\mathbb{R}} K(x,y) f(y) d\mu_\alpha(y).$$

Then A is a bounded linear operator on $L^2(\mathbb{R}, d\mu_\alpha)$ with

$$\|Af\|_{2,\alpha} \leq \sqrt{C_1 C_2} \|f\|_{2,\alpha}.$$

Bewijs. Let $u \in L^2(\mathbb{R}, d\mu_\alpha)$. Then using Hölder's inequality (2.22) for $p = q = 2$, we obtain

$$\begin{aligned} |Af(x)|^2 &\leq \left(\int_{\mathbb{R}} |K(x,y) f(y)| d\mu_\alpha(y) \right)^2 \\ &\leq \left(\int_{\mathbb{R}} |K(x,y)| d\mu_\alpha(y) \right) \left(\int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^2 d\mu_\alpha(y) \right) \\ &\leq C_1 \int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^2 d\mu_\alpha(y). \end{aligned}$$

Integrating with respect to variable x, we have

$$\|Af\|_{2,\alpha}^2 \leq C_1 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^2 d\mu_\alpha(y) \right) d\mu_\alpha(x) = C_1 C_2 \|f\|_{2,\alpha}^2.$$

□

Now, let us define convolution kernel. Thanks to the expression (3.9), we are able to calculate

$$K(x,y) = \int_{\mathbb{R}} E_\alpha(x,\lambda) E_\alpha(-y,\lambda) (1+\lambda^2)^{-\ell} a(x,\lambda) d\mu_\alpha(\lambda)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(z,\lambda)(1+\lambda^2)^{-\ell} a(x,\lambda) d\nu_{x,-y}(z) d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}} k(x,z) d\nu_{x,-y}(z),$$

where

$$k(x,z) = \int_{\mathbb{R}} E_\alpha(z,\lambda)(1+\lambda^2)^{-\ell} a(x,\lambda) d\mu_\alpha(\lambda) = \mathcal{F}_\alpha^{-1}[(1+\cdot^2)^{-\ell} a(x,\cdot)](z). \qquad (3.10)$$

The last integral exists, since $\ell > m + 2(\alpha+1)$.

According to previous calculations we can formally write pseudo-differential operators in various ways:

$$\begin{aligned} T_a f(x) &= \int_{\mathbb{R}} E_\alpha(x,\lambda) a(x,\lambda) \mathcal{F}_\alpha[f](\lambda) d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\lambda) E_\alpha(-y,\lambda) a(x,\lambda) f(y) d\mu_\alpha(y) d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(z,\lambda) a(x,\lambda) f(y) d\nu_{x,-y}(z) d\mu_\alpha(y) d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} k(x,z) f(y) d\nu_{x,-y}(z) d\mu_\alpha(y) \\ &= \int_{\mathbb{R}} K(x,y) f(y) d\mu_\alpha(y). \end{aligned}$$

Theorem 3.21 (Convolution kernel of a pseudo-differential operator). *Assume that $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$. Then convolution kernel*

$$k(x,z) = \int_{\mathbb{R}} E_\alpha(z,\lambda)(1+\lambda^2)^{-\ell} a(x,\lambda) d\mu_\alpha(\lambda)$$

of the pseudo-differential operator T_a satisfies

$$|\partial_x^s k(x,z)| \le \frac{C_{n,s}}{|z|^n}, \quad x, z \in \mathbb{R}, \quad \text{and} \quad z \neq 0$$

for $m + \delta s + 2(\alpha+1) < \ell + \rho n$.

Bewijs. Using integral representation of the convolution kernel we obtain

$$\begin{aligned} (iz)^n \partial_x^s k(x,z) &= \int_{\mathbb{R}} (iz)^n E_\alpha(z,\lambda)(1+\lambda^2)^{-\ell} \partial_x^s a(x,\lambda) d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} D^n_{\alpha,\lambda} E_\alpha(z,\lambda)(1+\lambda^2)^{-\ell} \partial_x^s a(x,\lambda) d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} E_\alpha(z,\lambda) D^n_{\alpha,\lambda} \left((1+\lambda^2)^{-\ell} \partial_x^s a(x,\lambda)\right) d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} E_\alpha(z,\lambda)(1+\lambda^2)^{-\ell} D^n_{\alpha,\lambda} \partial_x^s a(x,\lambda) d\mu_\alpha(\lambda), \end{aligned}$$

where $\ell > m + 2(\alpha+1)$.Hence, we have

$$|\partial_x^s k(x,z)| \le \frac{1}{|z|^n} \int_{\mathbb{R}} (1+\lambda^2)^{-\ell} \left| D^n_{\alpha,\lambda} \partial_x^s a(x,\lambda) \right| d\mu_\alpha(\lambda)$$

$$\leq \frac{C_{n,s}}{|z|^n} \int_{\mathbb{R}} \frac{(1+|\lambda|)^{m-\rho n+\delta s}}{(1+\lambda^2)^{\ell}} d\mu_\alpha(\lambda)$$
$$< +\infty$$

for $m + \delta k + 2(\alpha+1) < \ell + \rho n$. □

Proposition 3.22. *Let $f \in \mathcal{S}(\mathbb{R})$. Then for any continuous linear pseudo-differential operator*

$$T_a : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$$

with symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ there exists a unique convolution kernel $k \in \mathcal{S}'(\mathbb{R}\times\mathbb{R})$ such that

$$T_a f(x) = (k(x,\cdot) *_\alpha f)(x).$$

Bewijs. Let $f \in \mathcal{S}(\mathbb{R})$. Then rewriting the expression

$$T_a f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} k(x,z) f(y) d\nu_{x,-y}(z) d\mu_\alpha(y)$$

of the pseudo-differential operator T_a, we obtain

$$\begin{aligned} T_a f(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} k(x,z) f(y) d\nu_{x,-y}(z) d\mu_\alpha(y) \\ &= \int_{\mathbb{R}} \tau_x k(x,-y) f(y) d\mu_\alpha(y) \\ &= (k(x,\cdot) *_\alpha f)(x). \end{aligned}$$

After taking into account discussion in the beginning of this section about kernel of the operator and Schwartz kernel theorem I (Theorem 2.18) we can complete our proof. □

3.3. **Boundedness of pseudo-differential operators generated by the Dunkl operator.** In this section, we introduce a space $L(\mathbb{R}, d\mu_\alpha)$ by Definition 3.23 and obtain some boundedness results for pseudo-differential operators and comositiono of the pseudo-differential operators generated by the Dunkl operator in this space.

Definition 3.23. Let us define the space $L(\mathbb{R}, d\mu_\alpha)$, as following

$$L(\mathbb{R}, d\mu_\alpha) := \{f \in L^1(\mathbb{R}, d\mu_\alpha) \ : \ \mathcal{F}_\alpha[f] \in L^1(\mathbb{R}, d\mu_\alpha)\}$$

with norm

$$\|f\|_L := \|\mathcal{F}_\alpha[f]\|_{1,\alpha} = \int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)|\, d\mu_\alpha(\lambda). \tag{3.11}$$

Assumption 3.24. *We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is defined as:*

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi) V(\xi,\lambda) d\mu_\alpha(\xi), \tag{3.12}$$

where $V(\xi,\lambda)$ is a complex valued measurable function on $\mathbb{R}\times\mathbb{R}$, such that

$$|V(\xi,\lambda)| \leq K(\xi),$$

for all $\xi, \lambda \in \mathbb{R}$ and $K \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Remark 3.25. The integral (3.12) exists, because

$$|a(x,\lambda)| \leq \int_{\mathbb{R}} |E_\alpha(x,\xi)V(\xi,\lambda)|d\mu_\alpha(\xi) \leq \int_{\mathbb{R}} |K(\xi)|d\mu_\alpha(\xi) < +\infty.$$

Theorem 3.26. *Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator*

$$T_a f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)$$

is a bounded linear operator under Assumption 3.24 on $L(\mathbb{R}, d\mu_\alpha)$, i.e.

$$\|T_a f\|_L \leq 4\|K\|_{1,\alpha}\|f\|_L. \tag{3.13}$$

Bewijs. Let $f \in \mathcal{S}(\mathbb{R})$. Then by the definition of the pseudo-differential operator we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(x,\xi)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} E_\alpha(x,\eta)d\nu_{\lambda,\xi}(\eta)\right)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(\lambda,\xi,\eta)|\eta|^{2\alpha+1}d\eta d\mu_\alpha(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)|\xi|^{2\alpha+1}d\xi d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\eta)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\lambda)\right)d\mu_\alpha(\eta),
\end{aligned}$$

using above assumption and Fubini's theorem. After applying the Dunkl transform $\mathcal{F}_\alpha$ to the both sides of the equation we have

$$\begin{aligned}
\mathcal{F}_\alpha[T_a f](\eta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)|\xi|^{2\alpha+1}d\xi d\mu_\alpha(\lambda)
\end{aligned}$$

Hence, taking integral from both sides we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a f](\eta)|d\mu_\alpha(\eta)\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |V(\xi,\lambda)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)||\xi|^{2\alpha+1}d\xi d\mu_\alpha(\lambda)d\mu_\alpha(\eta)\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)||\xi|^{2\alpha+1}d\xi d\mu_\alpha(\lambda)d\mu_\alpha(\eta)
\end{aligned}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)W_\alpha(\lambda,\xi,\eta)||\eta|^{2\alpha+1}d\eta d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$\leq 4\int_{\mathbb{R}}\int_{\mathbb{R}} K(\xi)|\mathcal{F}_\alpha[f](\lambda)|d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$\leq 4\|K\|_{1,\alpha}\int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)|d\mu_\alpha(\lambda).$$

This completes proof of the theorem. □

Let $f, g \in \mathcal{S}(\mathbb{R})$. The composition of two pseudo-differential operators

$$T_a g(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)g(y)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$

and

$$T_b f(y) = \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(y,\xi)E_\alpha(-z,\xi)b(y,\xi)f(z)d\mu_\alpha(z)d\mu_\alpha(\xi),$$

with the symbols $a(x,\lambda)$ and $b(y,\xi)$ respectively, is

$$\begin{aligned} T_a(T_b f)(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)a(x,\lambda)E_\alpha(y,\xi)E_\alpha(-z,\xi)b(y,\xi)f(z) \\ &\quad \times d\mu_\alpha(z)d\mu_\alpha(\xi)d\mu_\alpha(y)d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\xi)E_\alpha(-z,\xi)c(x,\xi)f(z)d\mu_\alpha(z)d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}} E_\alpha(x,\xi)c(x,\xi)\mathcal{F}_\alpha[f](\xi)d\mu_\alpha(\xi), \end{aligned}$$

where

$$c(x,\xi) = \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi)d\mu_\alpha(y)d\mu_\alpha(\lambda).$$

Thus,

$$T_c f(x) = T_a(T_b f)(x) = \int_{\mathbb{R}} E_\alpha(x,\xi)c(x,\xi)\mathcal{F}_\alpha[f](\xi)d\mu_\alpha(\xi)$$

is a PDO with symbol

$$c(x,\xi) = \frac{1}{E_\alpha(x,\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi)d\mu_\alpha(y)d\mu_\alpha(\lambda).$$

Now, let us discuss about existence such an integral under Assumption 3.24. Let

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\eta)V_a(\eta,\lambda)d\mu_\alpha(\eta) \tag{3.14}$$

and

$$b(y,\xi) = \int_{\mathbb{R}} E_\alpha(y,\sigma)V_b(\sigma,\xi)d\mu_\alpha(\sigma), \tag{3.15}$$

where $V_a(\eta,\lambda)$ and $V_b(\sigma,\xi)$ are complex valued measurable functions on $\mathbb{R}\times\mathbb{R}$, such that

$$|V_a(\eta,\lambda)| \leq K_a(\eta) \quad \text{and} \quad |V_b(\sigma,\xi)| \leq K_b(\sigma)$$

for all $\eta, \lambda, \sigma, \xi \in \mathbb{R}$ and $K_a, K_b \in L^1(\mathbb{R}, d\mu_\alpha)$ are continuous functions. Then by using integral expressions (3.14) and (3.15) of $a(x,\lambda)$ and $b(y,\xi)$ respectively, we obtain

$$\begin{aligned}
c(x,\xi) &= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)a(x,\lambda)b(y,\xi)d\mu_\alpha(y)d\mu_\alpha(\lambda)\\
&= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)E_\alpha(y,\xi)E_\alpha(x,\eta)E_\alpha(y,\sigma)\\
&\times V_a(\eta,\lambda)V_b(\sigma,\xi)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(y)d\mu_\alpha(\lambda)\\
&= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(-y,\lambda)E_\alpha(y,\xi)E_\alpha(y,\sigma)d\mu_\alpha(y)\right)\\
&\times E_\alpha(x,\eta)V_a(\eta,\lambda)V_b(\sigma,\xi)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&= \frac{1}{E_\alpha(x,\xi)} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)W_\alpha(\xi,\sigma,\lambda)E_\alpha(x,\eta)V_a(\eta,\lambda)V_b(\sigma,\xi)\\
&\times d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda).
\end{aligned}$$

After taking absolute value from both sides of the equation, as following

$$\begin{aligned}
|c(x,\xi)| &\le \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |W_\alpha(\xi,\sigma,\lambda)V_a(\eta,\lambda)V_b(\sigma,\xi)|d\mu_\alpha(\sigma)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)\\
&\le 4\int_{\mathbb{R}}\int_{\mathbb{R}} K_a(\eta)K_b(\sigma)d\mu_\alpha(\sigma)d\mu_\alpha(\eta)\\
&\le 4\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha},
\end{aligned}$$

we can see that the $c(x,\xi)$ is a bounded function.

Corollary 3.27. *Let T_a and T_b are pseudo-differential operators with symbols a and b, respectively. Then under Assumption 3.24 their composition is a pseudo-differential operator $T_a \circ T_b$, which is continuous linear map on $\mathcal{S}(\mathbb{R})$.*

Corollary 3.28. *Let $f \in \mathcal{S}(\mathbb{R})$. Then under Assumption 3.24 the composition of pseudo-differential operators T_a and T_b is a bounded linear operator on $L(\mathbb{R}, d\mu_\alpha)$, i.e.*

$$\|T_a(T_bf)\|_L \le \frac{16}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha}\|f\|_L. \tag{3.16}$$

Bewijs. Let $f \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}
T_a(T_bf)(x) &= \int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V_a(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}_\alpha[T_a(T_bf)](\eta)\\
&= \int_{\mathbb{R}} E_\alpha(-x,\eta)T_a(T_bf)(x)d\mu_\alpha(x)\\
&= \int_{\mathbb{R}} E_\alpha(-x,\eta)\left(\int_{\mathbb{R}} E_\alpha(x,\lambda)\left(\int_{\mathbb{R}} E_\alpha(x,\xi)V_a(\xi,\lambda)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\lambda)\right)
\end{aligned}$$

$$\times\, d\mu_\alpha(x)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(-x,\eta)E_\alpha(x,\lambda)E_\alpha(x,\xi)V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)d\mu_\alpha(x)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} E_\alpha(-x,\eta)E_\alpha(x,\lambda)E_\alpha(x,\xi)d\mu_\alpha(x)\right)V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\int_{\mathbb{R}} W_\alpha(\lambda,\xi,\eta)V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$

where we have used (2.33). Then taking absolute value and integrating we have

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a(T_bf)](\eta)|\, d\mu_\alpha(\eta)$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} |W_\alpha(\lambda,\xi,\eta)V_a(\xi,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)|$$
$$\times\, d\mu_\alpha(\xi)d\mu_\alpha(\lambda)d\mu_\alpha(\eta)$$
$$\leq \frac{4}{2^{\alpha+1}\Gamma(\alpha+1)}\int_{\mathbb{R}}\int_{\mathbb{R}} K_a(\xi)\,|\mathcal{F}_\alpha[T_bf](\lambda)|\, d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$\leq \frac{4}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_bf](\lambda)|\, d\mu_\alpha(\lambda)$$
$$\leq \frac{16}{2^{\alpha+1}\Gamma(\alpha+1)}\|K_a\|_{1,\alpha}\|K_b\|_{1,\alpha}\int_{\mathbb{R}} |\mathcal{F}_\alpha[f](\lambda)|\, d\mu_\alpha(\lambda).$$

□

Assumption 3.29. *We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is defined by*

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V(\xi,\lambda)d\mu_\alpha(\xi),$$

satisfies

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)V_2(\lambda)d\mu_\alpha(\xi) = V_2(\lambda)\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi),$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function.

Theorem 3.30. *Let $f \in \mathcal{S}(\mathbb{R})$. Then the pseudo-differential operator T_a with symbol $a(x,\lambda)$, which satisfies Assumption 3.29, has a representation*

$$T_af(x) = 2^{\alpha+1}\Gamma(\alpha+1)\mathcal{F}_\alpha^{-1}(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(x)$$

and satisfies following inequality

$$\|T_af\|_L \leq 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha}. \tag{3.17}$$

Bewijs. By using Assumption 3.29 we have

$$\int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}} E_\alpha(x,\lambda)\left(V_2(\lambda)\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi)\right)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\lambda)E_\alpha(x,\xi)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\nu_{\lambda,\xi}(\eta)d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)W_\alpha(\lambda,\xi,\eta)|\eta|^{2\alpha+1}d\eta d\mu_\alpha(\xi)d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)W_\alpha(-\lambda,\eta,\xi)|\xi|^{2\alpha+1}d\xi d\mu_\alpha(\eta)d\mu_\alpha(\lambda)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} E_\alpha(x,\eta)V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\eta)d\mu_\alpha(\lambda)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(x,\eta)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} V_2(\lambda)V_1(\xi)\mathcal{F}_\alpha[f](\lambda)d\nu_{-\lambda,\eta}(\xi)d\mu_\alpha(\lambda)\right)d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(x,\eta)\left(\int_{\mathbb{R}} \tau_\eta V_1(-\lambda)V_2(\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)\right)d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} E_\alpha(x,\eta)(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(\eta)d\mu_\alpha(\eta)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1)\mathcal{F}_\alpha^{-1}(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(x).$$

Thus, applying the Dunkl transform we obtain

$$\mathcal{F}_\alpha[T_a f](\eta) = 2^{\alpha+1}\Gamma(\alpha+1)(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(\eta). \tag{3.18}$$

By taking integral from both sides of the above equation, we able to calculate

$$\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a f](\eta)|\, d\mu_\alpha(\eta) = 2^{\alpha+1}\Gamma(\alpha+1)\int_{\mathbb{R}} |(V_1 *_\alpha V_2\mathcal{F}_\alpha[f])(\eta)|\, d\mu_\alpha(\eta)$$

and

$$\|\mathcal{F}_\alpha[T_a f]\|_{1,\alpha} = 2^{\alpha+1}\Gamma(\alpha+1)\|V_1 *_\alpha V_2\mathcal{F}_\alpha[f]\|_{1,\alpha} \le 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where we have used the Proposition. Further by using the Definition of the Sobolev type space, it can be written as

$$\|T_a f\|_L \le 2^{\alpha+3}\Gamma(\alpha+1)\|V_1\|_{1,\alpha}\|V_2\mathcal{F}_\alpha[f]\|_{1,\alpha}.$$

□

Corollary 3.31. *Let*

$$a_k(x) = \int_{\mathbb{R}} E_\alpha(x,\xi)V_1^k(\xi)d\mu_\alpha(\xi),$$

where $V_1^k \in L^1(\mathbb{R}, d\mu_\alpha)$ *is a continuous function for all* k. *Then the operator*

$$\begin{cases} P_{n,\alpha} = \sum_{k=0}^n a_k(x)D_\alpha^k \\ Dom(P_{n,\alpha}) = \mathcal{S}(\mathbb{R}) \end{cases}$$

is a continuous linear operator from $\mathcal{S}(\mathbb{R})$ *to* $L(\mathbb{R}, d\mu_\alpha)$. *Moreover, we have*

$$\|P_{n,\alpha} f\|_L \le \sum_{k=0}^n 2^{\alpha+3}\Gamma(\alpha+1)\|V_1^k\|_{1,\alpha}\|V_2^k\mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where $V_2^k(\lambda) = (i\lambda)^k$.

Bewijs. Let $f \in \mathcal{S}(\mathbb{R})$. Then

$$f(x) = \int_{\mathbb{R}} E_\alpha(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda)$$

and

$$\begin{aligned} P_{n,\alpha}f(x) &= \sum_{k=0}^{n}\int_{\mathbb{R}} a_k(x)D_\alpha^k E_\alpha(x,\lambda)\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda) \\ &= \sum_{k=0}^{n}\int_{\mathbb{R}} E_\alpha(x,\lambda)a_k(x)(i\lambda)^k\mathcal{F}_\alpha[f](\lambda)d\mu_\alpha(\lambda). \end{aligned}$$

Hence, symbol of the pseudo-differential operator $P_{n,\alpha}$ expressed by the form

$$a(x,\lambda) = \sum_{k=0}^{n} a_k(x,\lambda) = \sum_{k=0}^{n} a_k(x)(i\lambda)^k = \sum_{k=0}^{n}(i\lambda)^k\int_{\mathbb{R}} E_\alpha(x,\xi)V_1^k(\xi)d\mu_\alpha(\xi).$$

Then by applying Theorem 3.30, we obtain

$$\|P_{n,\alpha}f\|_L \le \sum_{k=0}^{n} 2^{\alpha+3}\Gamma(\alpha+1)\|V_1^k\|_{1,\alpha}\|V_2^k\mathcal{F}_\alpha[f]\|_{1,\alpha},$$

where $V_2^k(\lambda) = (i\lambda)^k$. □

Assumption 3.32. *We assume the symbol $a \in S^m_{\rho,\delta}(\mathbb{R}\times\mathbb{R})$ is defined by*

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V(\xi,\lambda)d\mu_\alpha(\xi),$$

satisfies

$$a(x,\lambda) = \int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)V_2(\lambda)d\mu_\alpha(\xi) = V_2(\lambda)\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi),$$

where $V_1 \in L^1(\mathbb{R}, d\mu_\alpha)$ is a continuous function and $V_2(\lambda) = A$ is a constant. So we have

$$a(x,\lambda) = A\int_{\mathbb{R}} E_\alpha(x,\xi)V_1(\xi)d\mu_\alpha(\xi).$$

Theorem 3.33. *Let $f \in \mathcal{S}(\mathbb{R})$. Then the composition of the pseudo-differential operators T_a and T_b with symbols a and b, which satisfy Assumption 3.32, has a representation*

$$T_a(T_bf)(x) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 A\cdot\mathcal{F}_\alpha^{-1}[V_1 *_\alpha (W_1 *_\alpha B\cdot\mathcal{F}_\alpha[f])](x)$$

and satisfies following inequality

$$\|T_a(T_bf)\|_L \le 16\left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB\|V_1\|_{1,\alpha}\|W_1\|_{1,\alpha}\|f\|_L. \tag{3.19}$$

Bewijs. Let $f \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned} &T_a(T_bf)(x) \\ &= \int_{\mathbb{R}} E_\alpha(x,\lambda)a(x,\lambda)\mathcal{F}_\alpha[T_bf](\lambda)d\mu_\alpha(\lambda) \end{aligned}$$

$$= \int_{\mathbb{R}} E_\alpha(x,\lambda) \left(\int_{\mathbb{R}} E_\alpha(x,\xi) V_a(\xi,\lambda) d\mu_\alpha(\xi) \right) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\lambda)$$
$$= A \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\lambda) E_\alpha(x,\xi) V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\xi) d\mu_\alpha(\lambda)$$
$$= A \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\nu_{\lambda,\xi}(\eta) d\mu_\alpha(\xi) d\mu_\alpha(\lambda)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1) A \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha(x,\eta) V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\eta) d\mu_\alpha(\lambda)$$
$$= 2^{\alpha+1}\Gamma(\alpha+1) A \int_{\mathbb{R}} E_\alpha(x,\eta) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} V_1(\xi) \mathcal{F}_\alpha[T_b f](\lambda) d\nu_{-\lambda,\eta}(\xi) d\mu_\alpha(\lambda) \right) d\mu_\alpha(\eta).$$

Now an application of the Dunkl transform gives us

$$\begin{aligned} \mathcal{F}_\alpha[T_a(T_b f)](\eta) &= 2^{\alpha+1}\Gamma(\alpha+1) A \int_{\mathbb{R}} \tau_\eta V_1(-\lambda) \mathcal{F}_\alpha[T_b f](\lambda) d\mu_\alpha(\lambda) \\ &= 2^{\alpha+1}\Gamma(\alpha+1) A (V_1 *_\alpha \mathcal{F}_\alpha[T_b f])(\eta) \end{aligned}$$

Then by using (3.18) we obtain

$$\mathcal{F}_\alpha[T_a(T_b f)](\eta) = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 A (V_1 *_\alpha (W_1 *_\alpha B\mathcal{F}_\alpha[f]))(\eta),$$

so that

$$\begin{aligned} &\int_{\mathbb{R}} |\mathcal{F}_\alpha[T_a(T_b f)](\eta)| \, d\mu_\alpha(\eta) \\ &\qquad = \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \int_{\mathbb{R}} |(V_1 *_\alpha (W_1 *_\alpha \mathcal{F}_\alpha[f]))(\eta)| \, d\mu_\alpha(\eta). \end{aligned}$$

Thus we have

$$\begin{aligned} \|T_a(T_b f)\|_L &= \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1 *_\alpha (W_1 *_\alpha \mathcal{F}_\alpha[f])\|_{1,\alpha} \\ &\le 4 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1\|_{1,\alpha} \|W_1 *_\alpha \mathcal{F}_\alpha[f]\|_{1,\alpha} \\ &\le 16 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1\|_{1,\alpha} \|W_1\|_{1,\alpha} \|\mathcal{F}_\alpha[f]\|_{1,\alpha} \\ &= 16 \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^2 AB \|V_1\|_{1,\alpha} \|W_1\|_{1,\alpha} \|f\|_L. \end{aligned}$$

This completes proof of the theorem. □

4. Applications of Dunkl analysis

As an application of Dunkl analysis, we consider inverse source problems for time-fractional nonhomogeneous heat and pseudo-parabolic equations with Caputo fractional derivatives $\mathcal{D}_{0+}^{\gamma}$, $0<\gamma<1$, and bi-ordinal Hilfer fractional derivatives $D_{0+}^{(\gamma_1,\gamma_2)s}$, $0<\gamma_1,\gamma_2\leq 1$, $s\in[0,1]$, generated by the Dunkl operator D_α (2.8). The Section 4.1 deals with the inverse source problem for the time-fractional nonhomogeneous heat equation with Caputo fractional derivative, in Section 4.2, we study inverse source problem for the time-fractional nonhomogeneous pseudo-parabolic equation with Caputo fractional derivative, and in Section 4.3, we consider the time-fractional nonhomogeneous heat equation with bi-ordinal Hilfer fractional derivative.

Inverse source problem firstly was studied by W. Rundell and D. L. Colton in [69]. They considered the evolution type equation

$$\frac{du}{dt}+Au=f \tag{4.1}$$

in a Banach space X, where A is linear operator in X and f is a constant vector in X, with conditions

$$u(0)=u_0, \quad \text{and} \quad u(T)=u_1.$$

Using semigroups of operators Rundell proved a general theorem about the existence of a unique solution pair $(u(t),f)$ of the problem, which then was applied to equations of parabolic and pseudo-parabolic types. A.I. Prilepko and I.V. Tikhonov in their work [65] studied several inverse source problems for the equation (4.1) when the non-homogeneous term is represented in the form $f(t)=\Phi(t)f$, where $\Phi(t)$ is known operator and the element f is unknown, and A is a closed linear operator from $L_p(\Omega)$ into $L_p(\Omega)$ (Ω is some set). They applied obtained results to the transport equation. In [19] I. Bushuyev considered inverse source problems for the equation (4.1), where the unknown source depends on time, under a sufficient condition, with the linear elliptic partial differential operator A of order $2m$ with the bounded measurable coefficients such that

$$(A\varphi,\varphi)\geq\mu\|\varphi\|^2$$

for all $\varphi\in H^{2m}(\Omega)\cap H_0^m(\Omega)$, where μ is positive constant, and $(\cdot,\cdot)$ and $\|\cdot\|$ denote the standard scalar product and the norm in $L^2(\Omega)$ (Ω is a bounded domain in $\mathbb{R}^n$). When unknown source given by the general form $F(x,t)$ there is no closed theory. Known results deal with separated source terms. I.V. Tikhonov and Yu.S. Eidelman [89] considered inverse source problems for the generalization of the equation (4.1) of the form

$$\frac{d^N u(t)}{dt^N}=Au(t)+p, \quad 0<t<T,$$

for some positive integer $N\geq 1$ and some real number $T>0$ with an unknown parameter p and a closed linear operator A in the Banach space under the Cauchy conditions and över-determination condition"$u(T)=u_N$ (also in the Banach space). For the Laplace operator $(-\Delta)$ which is one of the most interesting examples in Physics, M. Choulli and M. Yamamoto in [23] established the uniqueness and conditional stability in determining a heat source term from boundary measurements with $f=\sigma(t)\varphi(x)$, where $\sigma(t)$ is known. M. Yaman and Ö. F. Gözükızıl in [97] studied asymptotic behaviour of the solution of the inverse source problem for the pseudo-parabolic equation

$$(u(x,t) - \Delta u(x,t))_t - \Delta u(x,t) + \alpha u(x,t) = f(t)g(x,t), \quad Q_\infty = \Omega \times (0,\infty)$$

with a integral over-determination condition.

Fractional derivatives and fractional partial differential equations have received great attention both in analysis and application, which are used in modeling several phenomena in different areas of science such as biology, physics, and chemistry, so the fractional computation is increasingly attracted to mathematicians in the last several decades. K. Sakamoto and M. Yamamoto in [78] considered inverse source problem for the time fractional parabolic equation

$$\mathcal{D}_t^\gamma u(x,t) = r^\gamma (Lu)(x,t) + f(x)h(x,t), \quad x \in \Omega, \ \ t \in (0,T), \ \ 0 < \gamma < 1,$$

where $\mathcal{D}_t^\gamma$ is the Caputo derivative defined by

$$\mathcal{D}_t^\gamma g(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \frac{d}{d\tau} g(\tau) d\tau$$

and L is a symmetric uniformly elliptic operator. The authors proved that the inverse problem is well-posed in the Hadamard sense except for a discrete set of values of diffusion constants using final overdetermining data. M. Yaman in [98] studied blow-up solution and stability to inverse source problem for the pseudo-parabolic equation

$$u_t - a\Delta u_t - \Delta u + \sum_{i=1}^n b_i u_{x_i} - |u|^p u = f(t)g(t), \quad x \in \Omega, t > 0$$

with the integral overdetermination condition. M. Slodička in [87] considered inverse source problem for the equation (4.1), when A is a linear differential operator of second-order, strongly elliptic, and the right-hand side f is assumed to be separable in both variables x and t, i.e. $f(x,t) = g(x)h(t)$ (in this case $h(t)$ is unknown). M. Slodička and K. Šišková in [85] studied inverse source problem for a semilinear time-fractional diffusion equation of second order in a bounded domain in $\mathbb{R}^d$

$$(g_{1-\beta} * \partial_t u(x))(t) + L(x,t)u(x,t) = h(t)f(x) + \int_0^t F(x,s,u(x,s))ds$$

with a linear second order differential operator $L(x,t)$ in the divergence form with space and time dependent coefficients. Authors showed the existence, uniqueness and regularity of a weak solution (u,h) ([85, Theorem 2.1, p. 1658]). Also, the inverse source problem for the heat equation

$$\mathcal{D}_t^\gamma u(x,t) = -\mathcal{L}u(x,t) + f(x)$$

with the Caputo fractional derivative $\mathcal{D}_t^\gamma$ was considered by M. Ruzhansky, N. Tokmagambetov, and B.T. Torebek in [70] in 2019, where $\mathcal{L}$ is a linear self-adjoint positive operator with a discrete spectrum $\{\lambda_\xi > 0 \ : \ \xi \in \mathcal{I}\}$ on a separable Hilbert space $\mathcal{H}$. Authors obtained unique solution pair (u,f) of the given equation under the conditions

$$u(x,0) = \varphi(x) \quad \text{and} \quad u(x,T) = \psi(x).$$

One of the recent papers for inverse source problems for pseudo-parabolic equations with fractional derivatives is [73]. In [73] M. Ruzhansky, D. Serikbaev, B.T. Torebek and N. Tokmagambetov have considered solvability of an inverse source problem for the pseudo-parabolic equation with the Caputo fractional derivative $\mathcal{D}_t^\gamma$ of order $0 < \gamma \le 1$

$$\mathcal{D}_t^\gamma (u(t) + \mathcal{L}u(t)) + \mathcal{M}u(t) = f(t) \quad \text{in} \quad \mathcal{H},$$
$$u(0) = \phi \in \mathcal{H}, \quad u(T) = \psi \in \mathcal{H},$$

where $\mathcal{H}$ be a separable Hilbert space and $\mathcal{L}$, $\mathcal{M}$ be operators with the corresponding discrete spectra on $\mathcal{H}$. The authors obtained well-posedness results.

A number of articles address the solvability of the inverse problems for parabolic ([3, 20, 83]), pseudo-parabolic ([7, 2, 50, 51, 52, 80, 54, 55]), and sub-diffusion equations ([21, 45, 47, 49, 61, 62]) and fractional diffusion equations ([82, 84, 90, 96]). We also would like to note recent works [8, 4, 27, 34, 43], where ISP was the subject of investigation.

An important motivation for studying non-local parabolic type problems for the Dunkl operators is related to their relevance for the evaluation analysis of many-body quantum systems of the Calogero-Moser-Sutherland type. These quantum systems describe algebraically integrable systems and are of considerable interest in mathematical physics, especially in con-formal field theory. For the related references we refer the reader to the book [95]. The semigroups $(H_t^{(\alpha,\beta)})_{t\ge 0}$ (the solution of the heat equation associated with the Jacobi-Dunkl operator $\Lambda^2_{\alpha,\beta}$) generate a new family of Markov processes on the real line. On some Riemannian symmetric spaces this process is the radial part of the Brownian motion for particular values of (α, β) [22].

4.1. **Time-fractional heat equation with Caputo fractional derivative.** In this section we prove the existence and uniqueness of the solution of the Cauchy problem

$$\begin{cases} \mathcal{D}^\gamma_{0+,t} u(t,x) - D^2_{\alpha,x} u(t,x) + m u(t,x) = f(t,x), & (t,x) \in Q_T, \\ u(0,x) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $Q_T := \{(t,x) : 0 < t < T, x \in \mathbb{R}\}$, $0 < \gamma < 1$, and m, T are given positive numbers and its limit case, when $\gamma = 1$,

$$\begin{cases} \partial_t u(t,x) - D^2_{\alpha,x} u(t,x) + m u(t,x) = f(t,x), & (t,x) \in Q_T, \\ u(0,x) = g(x), & x \in \mathbb{R}. \end{cases}$$

Then we study, the main problem of this section, the inverse source problem for the equation

$$\begin{cases} \mathcal{D}^\gamma_{0+,t} u(t,x) - D^2_{\alpha,x} u(t,x) + m u(t,x) = f(x), & (t,x) \in Q_T, \\ u(0,x) = \phi(x), & x \in \mathbb{R}, \\ u(T,x) = \psi(x), & x \in \mathbb{R}, \end{cases}$$

where $0 < \gamma < 1$ and its limit case, when $\gamma = 1$,

$$\begin{cases} \partial_t u(t,x) - D^2_{\alpha,x} u(t,x) + m u(t,x) = f(x), & (t,x) \in Q_T, \\ u(0,x) = \phi(x), & x \in \mathbb{R}, \\ u(T,x) = \psi(x), & x \in \mathbb{R}. \end{cases}$$

The existence and uniqueness results will be derived. Moreover, the stability theorem is also proved.

Remark 4.1. In Given problems, we impose the condition that m should be strictly positive to avoid technical issues when obtaining estimates. However, the problem can still be solved without this condition.

Remark 4.2. Results of this section are published in the "Journal of Inverse and Ill-Posed Problemsïn [10] in 2023(joint work with D. Serikbaev and N. Tokmagambetov).

Let us introduce the Sobolev space on $\mathbb{R}$, as following

$$\mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha) := \{f \in L^2(\mathbb{R}, d\mu_\alpha) : \quad (1+\lambda^2)\mathcal{F}_\alpha[f] \in L^2(\mathbb{R}, d\mu_\alpha)\}$$

with norm

$$\|f\|^2_{\mathcal{H}_\alpha} = \int_{\mathbb{R}} (1+\lambda^2)^2 |\mathcal{F}_\alpha[f](\lambda)|^2 d\mu_\alpha(\lambda).$$

4.1.1. *Direct problem for the time-fractional heat equation.* This subsection deals with the Cauchy problems for the nonhomogeneous heat equation with the Caputo fractional derivative $\mathcal{D}^\gamma_{0+}$, $0<\gamma<1$ and its limit case $\gamma = 1$, associated with the Dunkl operator (2.8).

Problem 4.3. *Let $0<\gamma<1$. Find the function u satisfying the equation*

$$\mathcal{D}^\gamma_{0+,t} u(t,x) - D^2_{\alpha,x} u(t,x) + m u(t,x) = f(t,x) \tag{4.2}$$

in the domain $(t,x) \in Q_T$, under the initial condition

$$u(0,x) = g(x), \quad x \in \mathbb{R}, \tag{4.3}$$

where f and g are sufficiently smooth functions, D_α is the Dunkl operator (2.8).

Definition 4.4. A generalised solution of Problem 4.3 is the function u from

$$C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$$

and satisfying the equation (4.2).

Theorem 4.5. *Let $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$, $f \in C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha))$ and $0<\gamma<1$. Then there exists a unique generalised solution of Problem 4.3. Moreover, it is given by the expression*

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}}\int_{\mathbb{R}} g(y) \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right) E_\alpha(x,\lambda) E_\alpha(-y,\lambda) d\mu_\alpha(y) d\mu_\alpha(\lambda) \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t f(\tau,y)(t-\tau)^{\gamma-1} \mathbb{E}_{\gamma,\gamma}\left(-(m+\lambda^2)(t-\tau)^\gamma\right) \\ &\qquad \times E_\alpha(x,\lambda) E_\alpha(-y,\lambda) d\tau d\mu_\alpha(y) d\mu_\alpha(\lambda), \end{aligned}$$

where E_α is the Dunkl kernel (2.12) and $\mathbb{E}_{\gamma,1}$ and $\mathbb{E}_{\gamma,\gamma}$ are Mittag-Leffler functions.

Bewijs. Let $0<\gamma<1$. We are looking for a solution of Problem 4.3 from $L^2(\mathbb{R}, d\mu_\alpha)$ and $f(t,\cdot) \in L^2(\mathbb{R}, d\mu_\alpha)$, so we able to apply the Dunkl transform $\mathcal{F}_\alpha$ (2.23) according to the x variable to the equation (4.2) and the initial condition (4.3). Then it gives us

$$\mathcal{D}^\gamma_{0+,t}\widehat{u}(t,\lambda) + (m+\lambda^2)\widehat{u}(t,\lambda) = \widehat{f}(t,\lambda), \tag{4.4}$$

and

$$\widehat{u}(0,\lambda) = \widehat{g}(\lambda), \tag{4.5}$$

for all $\lambda \in \mathbb{R}$, where $\widehat{u}(\cdot,\lambda)$ is an unknown function. Then by solving the equation (4.4) under the initial condition (4.5) (see [48, p. 231, ex. 4.9] and [53, p. 221]), we get

$$\widehat{u}(t,\lambda) = \widehat{g}(\lambda)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right) + \int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-(m+\lambda^2)(t-\tau)^\gamma\right)\widehat{f}(\tau,\lambda)d\tau, \tag{4.6}$$

where $\mathbb{E}_{\gamma,1}$ and $\mathbb{E}_{\gamma,\gamma}$ are Mittag-Leffler functions. Consequently, one obtains the solution of Problem 4.3, given by

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}}\int_{\mathbb{R}} g(y)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda) \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t f(\tau,y)(t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-(m+\lambda^2)(t-\tau)^\gamma\right) \\ &\qquad\times E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\tau d\mu_\alpha(y)d\mu_\alpha(\lambda) \end{aligned}$$

by using the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ (2.24) to (4.6), where E_α is the Dunkl kernel (2.12).

We have

$$(t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-(m+\lambda^2)(t-\tau)^\gamma\right) = \frac{1}{m+\lambda^2}\partial_\tau\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)(t-\tau)^\gamma\right).$$

Indeed, it follows from

$$\frac{d}{dx}\mathbb{E}_{\gamma,1}(x) = \frac{1}{\gamma}\mathbb{E}_{\gamma,\gamma}(x), \quad x \in \mathbb{R}.$$

Then, taking into account this and integrating by parts, one obtains

$$\begin{aligned} &\int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-(m+\lambda^2)(t-\tau)^\gamma\right)\widehat{f}(\tau,\lambda)d\tau \\ &= \frac{1}{m+\lambda^2}\int_0^t \partial_\tau\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)(t-\tau)^\gamma\right)\widehat{f}(\tau,\lambda)d\tau \\ &= \frac{\widehat{f}(t,\lambda)}{m+\lambda^2} - \frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)\widehat{f}(0,\lambda)}{m+\lambda^2} \\ &- \frac{1}{m+\lambda^2}\int_0^t \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)(t-\tau)^\gamma\right)\partial_\tau\widehat{f}(\tau,\lambda)d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} \widehat{u}(t,\lambda) = \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)\widehat{g}(\lambda) + \frac{\widehat{f}(t,\lambda)}{m+\lambda^2} - \frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)\widehat{f}(0,\lambda)}{m+\lambda^2} \\ - \frac{1}{m+\lambda^2}\int_0^t \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)(t-\tau)^\gamma\right)\partial_\tau\widehat{f}(\tau,\lambda)d\tau. \end{aligned}$$

Let us introduce following useful inequalities. For every $\lambda \in \mathbb{R}$, we have:

- if $m \geq 1$, then $0 < \frac{1+\lambda^2}{m+\lambda^2} \leq 1$ and $0 < \left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2 \leq 1$,

- if $0<m<1$, then $0<\frac{1+\lambda^2}{m+\lambda^2}<\frac{1}{m}$ and $0<\left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2<\frac{1}{m^2}$.

Let $g\in\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha)$ and $f\in C^1([0,T],L^2(\mathbb{R},d\mu_\alpha))$. Then for the function $u(t,\cdot)$ we have the following estimates

$$\begin{aligned}
\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha} &= \int_{\mathbb{R}}(1+\lambda^2)^2|\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda)\\
&\lesssim \int_{\mathbb{R}}(1+\lambda^2)^2\left|\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)\widehat{g}(\lambda)\right|^2 d\mu_\alpha(\lambda)\\
&+\int_{\mathbb{R}}\left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2\left|\widehat{f}(t,\lambda)\right|^2 d\mu_\alpha(\lambda)\\
&+\int_{\mathbb{R}}\left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2\left|\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)\widehat{f}(0,\lambda)\right|^2 d\mu_\alpha(\lambda)\\
&+\int_{\mathbb{R}}\left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2\left|\int_0^t\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)(t-\tau)^\gamma\right)\partial_\tau\widehat{f}(\tau,\lambda)d\tau\right|^2 d\mu_\alpha(\lambda)\\
&\lesssim \|g\|^2_{\mathcal{H}_\alpha}+\|\widehat{f}(t,\cdot)\|^2_{2,\alpha}+\|\widehat{f}(0,\cdot)\|^2_{2,\alpha}+\int_{\mathbb{R}}\left|\int_0^t\partial_\tau\widehat{f}(\tau,\lambda)d\tau\right|^2 d\mu_\alpha(\lambda)\\
&\lesssim \|g\|^2_{\mathcal{H}_\alpha}+\|\widehat{f}(t,\cdot)\|^2_{2,\alpha}+\|\widehat{f}(0,\cdot)\|^2_{2,\alpha}+\int_{\mathbb{R}}\int_0^t\left|\partial_\tau\widehat{f}(\tau,\lambda)\right|^2 d\tau d\mu_\alpha(\lambda)
\end{aligned}$$

where $U\lesssim W$ denotes $U\le CW$ for some positive constant C independent of U and W. Thus,

$$\begin{aligned}
\|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} &:= \max_{0\le t\le T}\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha}\\
&\lesssim \|g\|^2_{\mathcal{H}_\alpha}+\|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))}+\|\partial_t f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))}\\
&<+\infty.
\end{aligned}$$

and $u\in C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))$. Now, let us introduce also following useful inequalities. For every $\lambda\in\mathbb{R}$, we have:

- if $m>1$, then $0<m+\lambda^2<m(1+\lambda^2)$ and $0<(m+\lambda^2)^2<m^2(1+\lambda^2)^2$,
- if $0<m\le 1$, then $0<m+\lambda^2\le 1+\lambda^2$ and $0<(m+\lambda^2)^2\le(1+\lambda^2)^2$.

Then for $\mathcal{D}^\gamma_{0+,t}u$, we obtain

$$\begin{aligned}
\|\mathcal{D}^\gamma_{0+,t}u(t,\cdot)\|^2_{2,\alpha} &= \|\mathcal{F}_\alpha\left[\mathcal{D}^\gamma_{0+,t}u(t,\cdot)\right]\|^2_{2,\alpha}\\
&= \|\mathcal{D}^\gamma_{0+,t}\widehat{u}(t,\cdot)\|^2_{2,\alpha}\\
&= \int_{\mathbb{R}}\left|\widehat{f}(t,\lambda)-(m+\lambda^2)\widehat{u}(t,\lambda)\right|^2 d\mu_\alpha(\lambda)\\
&\lesssim \|\widehat{f}(t,\cdot)\|^2_{2,\alpha}+\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha}.
\end{aligned}$$

Consequently, it gives us

$$\|\mathcal{D}^\gamma_{0+,t}u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))}\lesssim\|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))}+\|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))}<+\infty.$$

Then using Definition 2.69, we obtain $u\in C^\gamma([0,T],L^2(\mathbb{R},d\mu_\alpha))$. The existence is proved.

Suppose that there are two solutions u_1 and u_2 of Problem 4.3. Denote

$$u(t, x) = u_1(t, x) - u_2(t, x).$$

Then the function u satisfies the equation

$$\mathcal{D}^{\gamma}_{0+,t} u(t, x) - D^2_{\alpha,x} u(t, x) + m \cdot u(t, x) = 0 \tag{4.7}$$

and the condition

$$u(0, x) = 0. \tag{4.8}$$

Then by applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23) to the equation (4.7) and the condition (4.8), one obtains

$$\mathcal{D}^{\gamma}_{0+,t} \widehat{u}(t, \lambda) + (m + \lambda^2)\widehat{u}(t, \lambda) = 0, \quad \widehat{u}(0, \lambda) = 0.$$

According to our analysis, above problem has a unique solution $\widehat{u}(t, \lambda) = 0$ for all $(t, \lambda) \in Q_T$. Hence, using Theorem 2.44 (Plancherel theorem) we obtain

$$0 = \|\widehat{u}\|_{2,\alpha} = \|u\|_{2,\alpha} \quad \text{and} \quad u(x, t) = u_1(t, x) - u_2(t, x) = 0$$

for all $(t, x) \in Q_T$. The uniqueness of the solution of Problem 4.3 is proved. □

Now, let us consider a limit case of Problem 4.3, when $\gamma = 1$. When $\gamma = 1$, instead of the Caputo fractional derivative we obtain usual partial derivative ∂_t. Then Problem 4.3 turns into the following problem:

Problem 4.6. *We aim to find a function u satisfying the equation*

$$\partial_t u(t, x) - D^2_\alpha u(t, x) + mu(t, x) = f(t, x), \quad (t, x) \in Q_T,$$

under the condition

$$u(0, x) = g(x), \quad x \in \mathbb{R}.$$

Theorem 4.7. *Let $f \in C^1([0, T], L^2(\mathbb{R}, d\mu_\alpha))$ and $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then Problem 4.6 has a unique generalised solution $u \in C^1([0, T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0, T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$ given by*

$$u(t, x) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(y) \exp(-(m + \lambda^2)t) E_\alpha(x, \lambda) E_\alpha(-y, \lambda) d\mu_\alpha(y) d\mu_\alpha(\lambda)$$
$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t f(\tau, y) \exp(-(m + \lambda^2)(t - \tau)) E_\alpha(x, \lambda) E_\alpha(-y, \lambda) d\tau d\mu_\alpha(y) d\mu_\alpha(\lambda). \tag{4.9}$$

Remark 4.8. The solution (4.9) agrees with the solution of Problem 4.3

$$u(t, x) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(y) \mathbb{E}_{\gamma,1} \left(-(m + \lambda^2)t^\gamma\right) E_\alpha(x, \lambda) E_\alpha(-y, \lambda) d\mu_\alpha(y) d\mu_\alpha(\lambda)$$
$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^t f(\tau, y)(t - \tau)^{\gamma-1} \mathbb{E}_{\gamma,\gamma} \left(-(m + \lambda^2)(t - \tau)^\gamma\right)$$
$$\times E_\alpha(x, \lambda) E_\alpha(-y, \lambda) d\tau d\mu_\alpha(y) d\mu_\alpha(\lambda).$$

We obtain (4.9), when $\gamma = 1$ (because $\mathbb{E}_{1,1}(x) = \exp(x)$).

Bewijs. Let us first prove the existence of the solution of Problem 4.6. By using the Dunkl transform $\mathcal{F}_\alpha$ (2.23) to Problem 4.6 according to the variable x, we obtain the ODE

$$\partial_t\widehat{u}(t,\lambda) + (m+\lambda^2)\widehat{u}(t,\lambda) = \widehat{f}(t,\lambda), \quad (t,\lambda)\in Q_T \tag{4.10}$$

with initial condition

$$\widehat{u}(0,\lambda) = \widehat{g}(\lambda), \quad \lambda\in\mathbb{R} \tag{4.11}$$

respect to the variable t. The general solution of the equation (4.10) can be written as

$$\widehat{u}(t,\lambda) = \int_0^t \widehat{f}(\tau,\lambda)\exp(-(m+\lambda^2)(t-\tau))d\tau + C(\lambda)\exp(-(m+\lambda^2)t), \tag{4.12}$$

where the function $C(\lambda)$ is unknown. After using (4.11), one has

$$\begin{aligned}
\widehat{u}(t,\lambda) &= \int_0^t \widehat{f}(\tau,\lambda)\exp(-(m+\lambda^2)(t-\tau))d\tau + \widehat{g}(\lambda)\exp(-(m+\lambda^2)t) \\
&= \frac{1}{m+\lambda^2}\int_0^t \widehat{f}(\tau,\lambda)\partial_\tau\exp(-(m+\lambda^2)(t-\tau))d\tau + \widehat{g}(\lambda)\exp(-(m+\lambda^2)t) \\
&= \frac{\widehat{f}(t,\lambda)}{m+\lambda^2} - \frac{\widehat{f}(0,\lambda)\exp(-(m+\lambda^2)t)}{m+\lambda^2} + \widehat{g}(\lambda)\exp(-(m+\lambda^2)t) \\
&\qquad - \frac{1}{m+\lambda^2}\int_0^t \partial_\tau\widehat{f}(\tau,\lambda)\exp(-(m+\lambda^2)(t-\tau))d\tau
\end{aligned}$$

Then applying the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ (2.24), we obtain

$$\begin{aligned}
u(t,x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} g(y)\exp(-(m+\lambda^2)t)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t f(\tau,y)\exp(-(m+\lambda^2)(t-\tau))E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\tau d\mu_\alpha(y)d\mu_\alpha(\lambda).
\end{aligned}$$

Let $f\in C^1([0,T],L^2(\mathbb{R},d\mu_\alpha))$ and $g\in\mathcal{H}_\alpha(\mathbb{R},\mu_\alpha)$. Then we have

$$\begin{aligned}
\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha} &= \int_{\mathbb{R}}(1+\lambda^2)^2|\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}}\left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2|\widehat{f}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}}\left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2|\widehat{f}(0,\lambda)\exp(-(m+\lambda^2)t)|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}}(1+\lambda^2)^2\,|\widehat{g}(\lambda)|^2\,d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}}\left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2\left|\int_0^t \partial_\tau\widehat{f}(\tau,\lambda)\exp(-(m+\lambda^2)(t-\tau))d\tau\right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\widehat{f}(t,\cdot)\|^2_{2,\alpha} + \|\widehat{f}(0,\cdot)\|^2_{2,\alpha} + \|g\|^2_{\mathcal{H}_\alpha} + \int_{\mathbb{R}}\left(\int_0^t|\partial_\tau\widehat{f}(\tau,\lambda)|d\tau\right)^2 d\mu_\alpha(\lambda)
\end{aligned}$$

Thus,

$$\|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} + \|\partial_t f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} + \|g\|^2_{\mathcal{H}_\alpha} < +\infty.$$

Now estimating the function u_t by

$$\begin{aligned}
\|\partial_t u(t,\cdot)\|^2_{2,\alpha} &= \|\partial_t \widehat{u}(t,\cdot)\|^2_{2,\alpha} \\
&= \int_{\mathbb{R}} |\partial_t \widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} |\widehat{f}(t,\lambda) - (m+\lambda^2)\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\widehat{f}(t,\cdot)\|^2_{2,\alpha} + \int_{\mathbb{R}} |(1+\lambda^2)\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda)
\end{aligned}$$

one gets

$$\|\partial_t u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} + \|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} < +\infty.$$

The existence is proved.

Now, we will prove the uniqueness of the solution of Problem 4.6. Let us suppose that u_1 and u_2 are two different solutions of Problem 4.6. Then $u(t,x) = u_1(t,x) - u_2(t,x)$ is the solution to the following problem:

$$u_t(t,x) - D^2_\alpha u(t,x) + mu(t,x) = 0, \quad (t,x) \in Q_T,$$

$$u(0,x) = 0, \quad x \in \mathbb{R}.$$

Then applying same technique, we able to see that the above problem has only trivial solution $u(t,x) = 0$ for all $(t,x) \in Q_T$, showing the uniqueness of the solutions of Problem 4.6. ☐

4.1.2. *Inverse source problems for the time-fractional heat equation.* In this subsection, we deal with a inverse source problem concerning the time-fractional heat equation with the Caputo fractional derivative $\mathcal{D}^\gamma_{0+}$, $0 < \gamma < 1$ generated by the Dunkl operator and its limit case $\gamma = 1$.

Problem 4.9. *Let $0 < \gamma < 1$. Find a pair of functions (u, f) satisfying the equation*

$$\mathcal{D}^\gamma_{0+,t} u(t,x) - D^2_{\alpha,x} u(t,x) + mu(t,x) = f(x) \tag{4.13}$$

in the domain $(t,x) \in Q_T$, under the initial condition

$$u(0,x) = \phi(x), \quad x \in \mathbb{R},$$

and the over-determination condition

$$u(T,x) = \psi(x), \quad x \in \mathbb{R},$$

where ϕ and ψ are sufficiently smooth functions, D_α is the Dunkl operator (2.8).

Definition 4.10. A generalised solution of Problem 4.9 is a pair of functions

$$u \in C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)) \quad \text{and} \quad f \in L^2(\mathbb{R}, d\mu_\alpha),$$

satisfying the equation (4.13).

Theorem 4.11. *Let $\psi, \phi \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$ and $0 < \gamma < 1$. Then a generalised solution of Problem 4.9 exists and is unique. Moreover, it can be written by the expressions*

$$u(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\phi(y) + \frac{\psi(y) - \phi(y)}{1 - \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} (1 - \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right) \right) \\ \times E_\alpha(x,\lambda) E_\alpha(-y,\lambda) d\mu_\alpha(y) d\mu_\alpha(\lambda)$$

and

$$f(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} (m+\lambda^2) \frac{\psi(y) - \phi(y)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \\ \times E_\alpha(x,\lambda) E_\alpha(-y,\lambda) d\mu_\alpha(y) d\mu_\alpha(\lambda),$$

where $\mathbb{E}_{\gamma,1}$ is the classical Mittag-Leffler function.

Bewijs. Here we want to find a generalised solution to (4.34). By applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23), according to the variable x, to the both sides, one obtains

$$\mathcal{D}^\gamma_{0+,t} \widehat{u}(t,\lambda) + (m+\lambda^2)\widehat{u}(t,\lambda) = \widehat{f}(\lambda), \quad (t,\lambda) \in Q_T, \tag{4.14}$$

$$\widehat{u}(0,\lambda) = \widehat{\phi}(\lambda), \quad \lambda \in \mathbb{R}, \tag{4.15}$$

$$\widehat{u}(T,\lambda) = \widehat{\psi}(\lambda), \quad \lambda \in \mathbb{R}, \tag{4.16}$$

where $\widehat{u}(\cdot,\lambda)$ and $\widehat{f}(\lambda)$ are unknown. The solutions of the equation (4.14) ([48]) are of the following form

$$\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{m+\lambda^2} + C(\lambda)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right), \tag{4.17}$$

where the constants $\widehat{f}(\lambda)$ and $C(\lambda)$ are unknown. To find these constants, we will use conditions (4.15) and (4.16). Hence, for $C(\lambda)$ we have

$$\widehat{u}(0,\lambda) = \frac{\widehat{f}(\lambda)}{m+\lambda^2} + C(\lambda) = \widehat{\phi}(\lambda),$$

$$\widehat{u}(T,\lambda) = \frac{\widehat{f}(\lambda)}{m+\lambda^2} + C(\lambda)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right) = \widehat{\psi}(\lambda),$$

$$\widehat{\phi}(\lambda) - C(\lambda) + C(\lambda)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right) = \widehat{\psi}(\lambda).$$

Thus,

$$C(\lambda) = \frac{\widehat{\phi}(\lambda) - \widehat{\psi}(\lambda)}{1 - \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}.$$

And, the unknown $\widehat{f}(\lambda)$ can by represented as

$$\widehat{f}(\lambda) = (m+\lambda^2)(\widehat{\phi}(\lambda) - C(\lambda)).$$

Consequently, by substituting $\widehat{f}(\lambda)$ and $C(\lambda)$ into (4.17), we arrive at

$$\widehat{u}(t,\lambda) = \widehat{\phi}(\lambda) + \frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)}{1 - \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \left(1 - \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)\right)$$

and

$$\widehat{f}(\lambda) = (m+\lambda^2) \frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}.$$

Finally, Problem 4.9 is formally solved and a pair of functions (u, f) are given by

$$u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\phi(y) + \frac{\psi(y)-\phi(y)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} (1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right) \right) \times E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$

and

$$f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} (m+\lambda^2) \frac{\psi(y)-\phi(y)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \times E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda),$$

by using the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ (2.24).

The inequalities (2.39) and (2.40) lead the following inequalities

$$0 < \frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} < \frac{1}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \leq 1 + \frac{1}{\Gamma(1+\gamma)^{-1}(m+\lambda^2)T^\gamma}.$$

Let $\psi, \phi \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$, then we have the following estimates

$$\begin{aligned}
\|f\|^2_{2,\alpha} = \|\widehat{f}\|^2_{2,\alpha} &= \int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} (m+\lambda^2)^2 \left| \frac{\widehat{\psi}(\lambda)-\widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}} (m+\lambda^2)^2 |\widehat{\psi}(\lambda)|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}} (m+\lambda^2)^2 |\widehat{\phi}(\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\psi\|^2_{\mathcal{H}_\alpha} + \|\phi\|^2_{\mathcal{H}_\alpha} < +\infty.
\end{aligned}$$

Thus, $f \in L^2(\mathbb{R}, d\mu_\alpha)$. For every fixed t we obtain

$$\begin{aligned}
\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha} &= \int_{\mathbb{R}} (1+\lambda^2)^2 |\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \lesssim \int_{\mathbb{R}} (1+\lambda^2)^2 \left|\widehat{\phi}(y)\right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{\widehat{\psi}(y)-\widehat{\phi}(y)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} (1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right) \right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\psi\|^2_{\mathcal{H}_\alpha} + \|\phi\|^2_{\mathcal{H}_\alpha} < +\infty.
\end{aligned}$$

Hence, it gives $u \in C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$. Rewriting the equation (4.14)

$$\mathcal{D}^\gamma_{0+,t}\widehat{u}(t,\lambda) = \widehat{f}(\lambda) - (m+\lambda^2)\widehat{u}(t,\lambda)$$

we arrive at

$$\begin{aligned}
\|\mathcal{D}^\gamma_{0+,t}u(t,\cdot)\|^2_{2,\alpha} &= \|\mathcal{F}_\alpha\left[\mathcal{D}^\gamma_{0+,t}u(t,\cdot)\right]\|^2_{2,\alpha} = \|\mathcal{D}^\gamma_{0+,t}\widehat{u}(t,\cdot)\|^2_{2,\alpha} \\
&= \int_{\mathbb{R}} |\widehat{f}(\lambda) - (m+\lambda^2)\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \lesssim \|f\|^2_{2,\alpha} + \|u(t,\cdot)\|^2_{\mathcal{H}_\alpha}.
\end{aligned}$$

Consequently, we obtain

$$\|\mathcal{D}^{\gamma}_{0+,t}u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{2,\alpha} + \|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} < +\infty$$

and $u \in C^{\gamma}([0,T], L^2(\mathbb{R}, d\mu_\alpha))$. The existence is proved.

Suppose that there are two solutions (u_1, f_1) and (u_2, f_2) of Problem 4.9. Denote

$$u(t,x) = u_1(t,x) - u_2(t,x)$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then the functions u and f satisfy the equation (4.13) and homogeneous conditions

$$u(0,x) = 0, \quad \text{and} \quad u(T,x) = 0. \tag{4.18}$$

Then by applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23) to the equation (4.13) and the conditions (4.18), we obtain

$$\mathcal{D}^{\gamma}_{0+,t}\widehat{u}(t,\lambda) + (m+\lambda^2)\widehat{u}(t,\lambda) = \widehat{f}(\lambda), \quad \widehat{u}(0,\lambda) = 0, \quad \widehat{u}(T,\lambda) = 0.$$

Consequently, we have $\widehat{u}(t,\lambda) = 0$, $\widehat{f}(\lambda) = 0$ for all $(t,\lambda) \in Q_T$, so Theorem 2.44 (Plancherel theorem) leads

$$0 = \|\widehat{u}\|_{2,\alpha} = \|u\|_{2,\alpha} \quad \text{and} \quad 0 = \|\widehat{f}\|_{2,\alpha} = \|f\|_{2,\alpha}.$$

Hence, $u(t,x) = u_1(t,x) - u_2(t,x) = 0$, $f(x) = f_1(x) - f_2(x) = 0$ for all $(t,x) \in Q_T$ and the uniqueness of the solutions of Problem 4.9 is proved. □

Now, let us consider limit case of Problem 4.9, when $\gamma = 1$. It is formulated as following:

Problem 4.12. *We aim to find a pair of functions (u, f) satisfying the equation*

$$\partial_t u(t,x) - D^2_\alpha u(t,x) + mu(t,x) = f(x), \quad (t,x) \in Q_T,$$

under the initial condition

$$u(0,x) = \phi(x), \quad x \in \mathbb{R},$$

and the over-determination condition

$$u(T,x) = \psi(x), \quad x \in \mathbb{R}$$

where ϕ and ψ are sufficiently smooth functions, D_α is the Dunkl operator (2.8).

Theorem 4.13. *Assume that $\phi, \psi \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then Problem 4.12 has a unique generalised solution (u, f), where $u \in C^1([0,T], L^2((\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$ and $f \in L^2(\mathbb{R}, d\mu_\alpha)$. Moreover, they can be represented in the forms*

$$u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1-\exp(-(m+\lambda^2)t)}{1-\exp(-(m+\lambda^2)T)}\psi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\exp(-(m+\lambda^2)t)-\exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)}\phi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$

and

$$f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}}(m+\lambda^2)\frac{\psi(y)-\phi(y)\exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)}E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda).$$

Remark 4.14. The solution of Problem 4.9 agrees with the solution of Problem 4.12, when $\gamma = 1$.

Bewijs. Let us first prove the existence part. Applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23), according to the variable x, to Problem 4.12, we obtain

$$\partial_t \widehat{u}(t,\lambda) + (m+\lambda^2)\widehat{u}(t,\lambda) = \widehat{f}(\lambda), \quad (t,\lambda) \in Q_T, \tag{4.19}$$

$$\widehat{u}(0,\lambda) = \widehat{\phi}(\lambda), \ \lambda \in \mathbb{R}, \tag{4.20}$$

$$\widehat{u}(T,\lambda) = \widehat{\psi}(\lambda), \ \lambda \in \mathbb{R}. \tag{4.21}$$

Then for every $\lambda \in \mathbb{R}$ the general solution of ordinary differential equation (4.19) is

$$\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{m+\lambda^2}(1-\exp(-(m+\lambda^2)t)) + C(\lambda)\exp(-(m+\lambda^2)t), \tag{4.22}$$

where the functions $C(\lambda)$ and $\widehat{f}(\lambda)$ are unknown. By using the conditions (4.20) and (4.21), one can find

$$\widehat{u}(0,\lambda) = C(\lambda) = \widehat{\phi}(\lambda),$$

$$\widehat{u}(T,\lambda) = \frac{\widehat{f}(\lambda)}{m+\lambda^2}(1-\exp(-(m+\lambda^2)T)) + \widehat{\phi}(\lambda)\exp(-(m+\lambda^2)T) = \widehat{\psi}(\lambda).$$

Then $\widehat{f}(\lambda)$ can be represented as

$$\widehat{f}(\lambda) = (m+\lambda^2)\frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)\exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)}. \tag{4.23}$$

Now, substituting the functions $C(\lambda)$ and $\widehat{f}(\lambda)$ into (4.22), one has

$$\widehat{u}(t,\lambda) = \frac{1-\exp(-(m+\lambda^2)t)}{1-\exp(-(m+\lambda^2)T)}\widehat{\psi}(\lambda) + \frac{\exp(-(m+\lambda^2)t) - \exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)}\widehat{\phi}(\lambda). \tag{4.24}$$

Finally, by using the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ (2.24) to (4.23) and (4.24), we obtain the solution to Problem 4.12:

$$u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1-\exp(-(m+\lambda^2)t)}{1-\exp(-(m+\lambda^2)T)}\psi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$
$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\exp(-(m+\lambda^2)t) - \exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)}\phi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda) \tag{4.25}$$

and

$$f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} (m+\lambda^2)\frac{\psi(y) - \phi(y)\exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)}E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda). \tag{4.26}$$

A simple calculations give us the following helpful inequalities:

- $0 \le \frac{1-\exp(-mt)}{1-\exp(-(m+\lambda^2)T)} \le \frac{1-\exp(-(m+\lambda^2)t)}{1-\exp(-(m+\lambda^2)T)} \le 1$,
- $0 \le \frac{\exp(-(m+\lambda^2)t) - \exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)} \le 1$,
- $1 < \frac{1}{1-\exp(-(m+\lambda^2)T)} < \frac{1}{1-\exp(-mT)}$,
- $1 < \frac{\exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)} < \frac{\exp(-mT)}{1-\exp(-mT)}$.

Let $\phi, \psi \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then for every fixed t, one can be obtained

$$\begin{aligned}
\|u(t,\cdot)\|^2_{W^{2,2}_\alpha} &= \int_{\mathbb{R}} (1+\lambda^2)^2 |\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{1-\exp(-(m+\lambda^2)t)}{1-\exp(-(m+\lambda^2)T)} \widehat{\psi}(\lambda) \right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{\exp(-(m+\lambda^2)t) - \exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)} \widehat{\phi}(\lambda) \right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\psi\|^2_{\mathcal{H}_\alpha} + \|\phi\|^2_{\mathcal{H}_\alpha} < +\infty.
\end{aligned}$$

Thus, we arrive at $u \in C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$. We also have

$$\begin{aligned}
\|f\|^2_{2,\alpha} = \|\widehat{f}\|^2_{2,\alpha} &= \int_{\mathbb{R}} |\widehat{f}(\lambda)|^2 d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} (m+\lambda^2)^2 \left| \frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)\exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)} \right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}} (m+\lambda^2)^2 \left| \frac{\widehat{\psi}(\lambda)}{1-\exp(-(m+\lambda^2)T)} \right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} (m+\lambda^2)^2 \left| \frac{\exp(-(m+\lambda^2)T)}{1-\exp(-(m+\lambda^2)T)} \widehat{\phi}(\lambda) \right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\psi\|^2_{\mathcal{H}_\alpha} + \|\phi\|^2_{\mathcal{H}_\alpha} < +\infty
\end{aligned}$$

and $f \in L^2(\mathbb{R}, d\mu_\alpha)$. Rewriting the equation (4.19), we get

$$\begin{aligned}
\|\partial_t u(t,\cdot)\|^2_{2,\alpha} = \|\widehat{\partial_t u}(t,\cdot)\|^2_{2,\alpha} &= \int_{\mathbb{R}} |\partial_t \widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&= \int_{\mathbb{R}} |\widehat{f}(\lambda) - (m+\lambda^2)\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\widehat{f}\|^2_{2,\alpha} + \int_{\mathbb{R}} |(m+\lambda^2)\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|f\|^2_{2,\alpha} + \|u(t,\cdot)\|^2_{\mathcal{H}_\alpha}.
\end{aligned}$$

Hence, we obtain

$$\|\partial_t u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{2,\alpha} + \|u\|^2_{C([0,T],W^{2,2}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|\psi\|_{\mathcal{H}_\alpha} + \|\phi\|_{\mathcal{H}_\alpha} < +\infty.$$

The existence is proved.

The uniqueness of the solutions of Problem 4.12 can be shown by taking into account the property of the Dunkl transform (Plancherel Theorem 2.44) and by seeing that the pair of functions (u, f) can be uniquely determined by the formulas the (4.25) and (4.26). □

4.1.3. *Stability.* In the subsections above we showed the uniqueness of the inverse source Problems 4.9 and 4.12. These kind of equations usually ill-posed. Hence, it is sensitive to the change of data. Practically, our final time measurement contains errors. In the following statement we address the impact of this on a solution of Problem 4.9. The case of Problem 4.12 can be dealt in a similar way.

Theorem 4.15. *Let (u, f) and (u_d, f_d) be solutions to Problem 4.9 corresponding to the data (ϕ, ψ) and its small perturbation (ϕ_d, ψ_d), respectively. Then the solution of Problem 4.9 depends continuously on these data, namely, we have*

$$\|u-u_d\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|\psi-\psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi-\phi_d\|^2_{\mathcal{H}_\alpha}$$

and

$$\|f-f_d\|^2_{2,\alpha} \lesssim \|\psi-\psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi-\phi_d\|^2_{\mathcal{H}_\alpha}.$$

Bewijs. From the definition of the Dunkl transform

$$\widehat{u}(t,\lambda) = \mathcal{F}_\alpha[u(t,\cdot)](\lambda) = \int_{\mathbb{R}} u(t,x)E_\alpha(-x,\lambda)d\mu_\alpha(x),$$

we have

$$\begin{aligned}\mathcal{F}_\alpha[u(t,\cdot)-u_d(t,\cdot)](\lambda) &= \int_{\mathbb{R}} (u(t,x)-u_d(t,x))E_\alpha(-x,\lambda)d\mu_\alpha(x)\\ &= \int_{\mathbb{R}} u(t,x)E_\alpha(-x,\lambda)d\mu_\alpha(x) - \int_{\mathbb{R}} u_d(t,x)E_\alpha(-x,\lambda)d\mu_\alpha(x)\\ &= \mathcal{F}_\alpha[u(t,\cdot)](\lambda) - \mathcal{F}_\alpha[u_d(t,\cdot)](\lambda)\\ &= \widehat{u}(t,\lambda) - \widehat{u}_d(t,\lambda),\end{aligned}$$

here we have used property of the integral. Then we arrive at

$$\|u(t,\cdot)-u_d(t,\cdot)\|^2_{\mathcal{H}_\alpha} = \int_{\mathbb{R}} (1+\lambda^2)^2|\widehat{u}(t,\lambda)-\widehat{u}_d(t,\lambda)|^2 d\mu_\alpha(\lambda)$$

$$\begin{aligned}&= \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\psi}(\lambda)\right.\\ &- \frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\phi}(\lambda)\\ &- \left(\frac{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\psi}_d(\lambda)\right.\\ &- \left.\left.\frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\phi}_d(\lambda)\right)\right|^2 d\mu_\alpha(\lambda)\\ &= \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\left(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\right)\right.\\ &- \left.\frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\left(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\right)\right|^2 d\mu_\alpha(\lambda)\\ &\lesssim \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\left(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\right)\right|^2 d\mu_\alpha(\lambda)\\ &+ \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\left(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\right)\right|^2 d\mu_\alpha(\lambda)\end{aligned}$$

$$\lesssim \int_{\mathbb{R}} (1+\lambda^2)^2 \left|\mathcal{F}_\alpha[\psi-\psi_d](\lambda)\right|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}} (1+\lambda^2)^2 \left|\mathcal{F}_\alpha[\phi-\phi_d](\lambda)\right|^2 d\mu_\alpha(\lambda)$$
$$\lesssim \|\psi-\psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi-\phi_d\|^2_{\mathcal{H}_\alpha}.$$

Consequently, we have

$$\|u(t,\cdot)-u_d(t,\cdot)\|^2_{\mathcal{H}_\alpha} \lesssim \|\psi-\psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi-\phi_d\|^2_{\mathcal{H}_\alpha},$$

or

$$\|u-u_d\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|\psi-\psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi-\phi_d\|^2_{\mathcal{H}_\alpha}.$$

Writing $\widehat{f}(\lambda)$ in the form

$$\widehat{f}(\lambda) = \frac{(m+\lambda^2)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\psi}(\lambda) - \frac{(m+\lambda^2)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\phi}(\lambda),$$

we obtain

$$\|f-f_d\|^2_{2,\alpha} = \|\mathcal{F}_\alpha[f-f_d]\|^2_{2,\alpha} = \|\widehat{f}-\widehat{f}_d\|^2_{2,\alpha} = \int_{\mathbb{R}} |\widehat{f}(\lambda)-\widehat{f}_d(\lambda)|^2 d\mu_\alpha(\lambda)$$

$$= \int_{\mathbb{R}} \Bigg| \frac{(m+\lambda^2)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\psi}(\lambda) - \frac{(m+\lambda^2)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\phi}(\lambda)$$
$$- \left(\frac{(m+\lambda^2)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\psi}_d(\lambda) - \frac{(m+\lambda^2)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}\widehat{\phi}_d(\lambda) \right) \Bigg|^2 d\mu_\alpha(\lambda)$$
$$= \int_{\mathbb{R}} \Bigg| \frac{(m+\lambda^2)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \left(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\right)$$
$$- \frac{(m+\lambda^2)\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \left(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\right) \Bigg|^2 d\mu_\alpha(\lambda)$$
$$\lesssim \int_{\mathbb{R}} (m+\lambda^2)^2 \left| \frac{1}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \left(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\right) \right|^2 d\mu_\alpha(\lambda)$$
$$+ \int_{\mathbb{R}} (m+\lambda^2)^2 \left| \frac{\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-(m+\lambda^2)T^\gamma\right)} \left(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\right) \right|^2 d\mu_\alpha(\lambda)$$
$$\lesssim \int_{\mathbb{R}} (m+\lambda^2)^2 \left|\mathcal{F}_\alpha[\psi-\psi_d](\lambda)\right|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}} (m+\lambda^2)^2 \left|\mathcal{F}_\alpha[\phi-\phi_d](\lambda)\right|^2 d\mu_\alpha(\lambda)$$
$$\lesssim \|\psi-\psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi-\phi_d\|^2_{\mathcal{H}_\alpha}.$$

Thus,

$$\|f-f_d\|^2_{2,\alpha} \lesssim \|\psi-\psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi-\phi_d\|^2_{\mathcal{H}_\alpha}.$$

It completes the proof. □

4.2. **Time-fractional pseudo-parabolic equation with Caputo fractional derivative.** In this section, we are interested in studying the Cauchy problem for the the time-fractional pseudo-parabolic equation

$$\begin{cases} \mathbb{D}^\gamma_{0^+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(t,x), \ (t,x) \in Q_T, \\ u(0,x) = g(x), \quad x \in \mathbb{R} \end{cases}$$

and the inverse source problem for the time-fractional pseudo-parabolic equation

$$\begin{cases} \mathbb{D}^{\gamma}_{0+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), \ (t,x) \in Q_T, \\ u(0,x) = \phi(x), \quad x \in \mathbb{R}, \\ u(T,x) = \psi(x), \quad x \in \mathbb{R}, \end{cases} \tag{4.27}$$

associated with the Dunkl operator D_α (2.8), where $a, m > 0$, and

$$\mathbb{D}^{\gamma}_{0+,t} = \begin{cases} \mathcal{D}^{\gamma}_{0+,t} & \text{if} \quad 0 < \gamma < 1, \\ \partial_t & \text{if} \quad \gamma = 1. \end{cases}$$

Remark 4.16. In the case $a = 0$, the inverse source problem for time-fractional pseudo-parabolic equation (4.27) reduces to the inverse source problem for the time-fractional heat equation

$$\begin{cases} \mathbb{D}^{\gamma}_{0+,t}u(t,x) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), \quad (t,x) \in Q_T, \\ u(0,x) = \phi(x), \quad x \in \mathbb{R}, \\ u(T,x) = \psi(x), \quad x \in \mathbb{R}, \end{cases}$$

which was considered in Section 4.1. So, in this paper we are interested considering only case when $a > 0$.

Remark 4.17. In Given problems, we impose the condition that m should be strictly positive to avoid technical issues when obtaining estimates. However, the problem can still be solved without this condition.

Remark 4.18. Results of this section are published as a preprint in ärxivïn [9] in 2023 (joint work with N. Tokmagambetov).

4.2.1. *Direct problem for the time-fractional pseudo-parabolic equation.* Here we consider the direct problem stated as following.

Problem 4.19. *Let $0 < \gamma \leq 1$. Our aim is to find the function u satisfying the equation*

$$\mathbb{D}^{\gamma}_{0+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(t,x), \quad (t,x) \in Q_T, \tag{4.28}$$

under the initial condition

$$u(0,x) = g(x), \quad x \in \mathbb{R}, \tag{4.29}$$

where f and g are sufficiently smooth functions.

Definition 4.20. A generalised solution of Problem 4.19 is a function u from

$$C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha))$$

and satisfying the equation (4.28).

Theorem 4.21. *a) Let $0 < \gamma < 1$. Assume that $f \in C^1([0,T], L^2(\mathbb{R}, d\mu_\alpha))$ and $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then Problem 4.19 has a generalised unique solution, which is given by the expression*

$$u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right) g(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$

$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma}\right)\frac{f(\tau,y)}{1+a\lambda^2} \times E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\tau d\mu_\alpha(y)d\mu_\alpha(\lambda), \quad (4.30)$$

where $\mathbb{E}_{\gamma,1}$ and $\mathbb{E}_{\gamma,\gamma}$ are the Mittag-Leffler functions.
b) Let $\gamma = 1$. Assume that $f \in C([0,T], L^2(\mathbb{R}, d\mu_\alpha))$ and $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$. Then Problem 4.19 has a unique generalised solution, which is given by the expression (4.30).

Bewijs. Solution of Problem 4.19 can be found by applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23) to the equation (4.28) and the initial condition (4.29). Thus, we have

$$\mathbb{D}^{\gamma}_{0^+,t}\widehat{u}(t,\lambda) + \frac{m+\lambda^2}{1+a\lambda^2}\widehat{u}(t,\lambda) = \frac{\widehat{f}(t,\lambda)}{1+a\lambda^2}, \quad (t,\lambda) \in Q_T, \quad (4.31)$$

and

$$\widehat{u}(0,\lambda) = \widehat{g}(\lambda), \quad \lambda \in \mathbb{R}, \quad (4.32)$$

where $\widehat{u}(\cdot,\lambda)$ is an unknown function. Let $0 < \gamma \leq 1$. For every fixed $\lambda \in \mathbb{R}$ the equation (4.31) is a ordinary differential equation, respect to the variable t, then by solving the equation (4.31) under the initial condition (4.32) (see [48, p. 231, ex. 4.9]), we obtain

$$\widehat{u}(t,\lambda) = \widehat{g}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^{\gamma}\right) + \int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma}\right)\frac{\widehat{f}(\tau,\lambda)}{1+a\lambda^2}d\tau, \quad (4.33)$$

where $\mathbb{E}_{\gamma,1}$ and $\mathbb{E}_{\gamma,\gamma}$ are the Mittag-Leffler functions. Consequently, solution of Problem 4.19 is

$$u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^{\gamma}\right) g(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma}\right)\frac{f(\tau,y)}{1+a\lambda^2} \times E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\tau d\mu_\alpha(y)d\mu_\alpha(\lambda),$$

here we have used the Fubini's theorem and the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ (2.24) to the expression (4.33).

By using the property

$$\frac{d}{d\tau}\left(\mathbb{E}_{\gamma,1}(c\tau^{\gamma})\right) = c\tau^{\gamma-1}\mathbb{E}_{\gamma,\gamma}(c\tau^{\gamma}), \quad c = \text{constant},$$

of the Mittag-Leffler function, we obtain

$$\partial_\tau\left(\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma}\right)\right) = \frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma}\right)$$

and

$$\widehat{g}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^{\gamma}\right) + \int_0^t (t-\tau)^{\gamma-1}\mathbb{E}_{\gamma,\gamma}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma}\right)\frac{\widehat{f}(\tau,\lambda)}{1+a\lambda^2}d\tau$$
$$= \widehat{g}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^{\gamma}\right) + \frac{1}{m+\lambda^2}\int_0^t \partial_\tau\left(\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^{\gamma}\right)\right)\widehat{f}(\tau,\lambda)d\tau$$

$$= \widehat{g}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right) + \frac{\widehat{f}(t,\lambda)}{m+\lambda^2} - \frac{1}{m+\lambda^2}\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)\widehat{f}(0,\lambda)$$
$$- \frac{1}{m+\lambda^2}\int_0^t \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^\gamma\right)\partial_\tau \widehat{f}(\tau,\lambda)d\tau$$

by using the Integration by Parts and the fact $\mathbb{E}_{\gamma,1}(0) = 1$.

We assume that $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$ and $f \in C^1([0,T], L^2(\mathbb{R}, d\mu_\alpha))$. Then for the function $u(t,\cdot)$ we have the following estimates

$$\begin{aligned}
\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha} &= \int_{\mathbb{R}} (1+\lambda^2)^2 |\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}} (1+\lambda^2)^2 \left|\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)\widehat{g}(\lambda)\right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} \left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2 \left|\widehat{f}(t,\lambda)\right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} \left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2 \left|\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)\widehat{f}(0,\lambda)\right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} \left(\frac{1+\lambda^2}{m+\lambda^2}\right)^2 \left|\int_0^t \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)^\gamma\right)\partial_t \widehat{f}(\tau,\lambda)d\tau\right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|g\|^2_{\mathcal{H}_\alpha} + \|\widehat{f}(t,\cdot)\|^2_{2,\alpha} + \|\widehat{f}(0,\cdot)\|^2_{2,\alpha} + \int_{\mathbb{R}} \left(\int_0^t |\partial_t \widehat{f}(\tau,\lambda)|d\tau\right)^2 d\mu_\alpha(\lambda)
\end{aligned}$$

Thus, we obtain

$$\|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|g\|^2_{\mathcal{H}_\alpha} + \|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} + \|\partial_t f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} < +\infty.$$

Now, for $\mathcal{D}^\gamma_{0+,t}u(t,\cdot)$ we have

$$\begin{aligned}
\|\mathcal{D}^\gamma_{0+,t}u(t,\cdot)\|^2_{2,\alpha} &= \|\mathcal{F}_\alpha\left[\mathcal{D}^\gamma_{0+,t}u(t,\cdot)\right]\|^2_{2,\alpha} \\
&= \|\mathcal{D}^\gamma_{0+,t}\widehat{u}(t,\cdot)\|^2_{2,\alpha} \\
&= \int_{\mathbb{R}} \left|\frac{\widehat{f}(t,\lambda)}{1+a\lambda^2} - \frac{m+\lambda^2}{1+a\lambda^2}\widehat{u}(t,\lambda)\right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}} \frac{1}{(1+a\lambda^2)^2}\left|\widehat{f}(t,\lambda)\right|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}} \left(\frac{m+\lambda^2}{1+a\lambda^2}\right)^2 |\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|f(t,\cdot)\|^2_{2,\alpha} + \|u(t,\cdot)\|^2_{\mathcal{H}_\alpha}.
\end{aligned}$$

Consequently, it gives us

$$\|\mathcal{D}^\gamma_{0+,t}u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} + \|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} < +\infty.$$

Finally, using Definition 2.69 we obtain $u \in C^\gamma([0,T], L^2(\mathbb{R}, d\mu_\alpha))$.

Let $\gamma = 1$. We assume that $g \in \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)$ and $f \in C([0,T], L^2(\mathbb{R}, d\mu_\alpha))$. Then let us estimate the function $u(t,\cdot)$ as follows

$$\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha} = \int_{\mathbb{R}} |(1+\lambda^2)\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda)$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}} \left(\frac{1+\lambda^2}{1+a\lambda^2}\right)^2 \left|\int_0^t \widehat{f}(\tau,\lambda)\exp\left(-\frac{m+\lambda^2}{1+a\lambda^2}(t-\tau)\right)d\tau\right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} (1+\lambda^2)^2 \left|\exp\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)\widehat{g}(\lambda)\right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}} \left(\int_0^t |\widehat{f}(\tau,\lambda)|d\tau\right)^2 d\mu_\alpha(\lambda) + \|g\|^2_{\mathcal{H}_\alpha}.
\end{aligned}
$$

Then

$$
\|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} + \|g\|^2_{\mathcal{H}_\alpha} < +\infty.
$$

Let us estimate $\partial_t u(t,\cdot)$ as follows

$$
\begin{aligned}
\|\partial_t u(t,\cdot)\|^2_{2,\alpha} &= \|\widehat{\partial_t u}(t,\cdot)\|^2_{2,\alpha} \\
&= \|\partial_t \widehat{u}(t,\cdot)\|^2_{2,\alpha} \\
&= \int_{\mathbb{R}} |\partial_t \widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \int_{\mathbb{R}} \frac{1}{(1+a\lambda^2)^2}\left|\widehat{f}(t,\lambda)\right|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}} \left(\frac{m+\lambda^2}{1+a\lambda^2}\right)^2 |\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|f(t,\cdot)\|^2_{2,\alpha} + \|u(t,\cdot)\|^2_{\mathcal{H}_\alpha}.
\end{aligned}
$$

Thus,

$$
\|\partial_t u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} + \|u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} < +\infty.
$$

The existence is proved.

Now, we are going to prove uniqueness of the solution. Suppose that there are two solutions u_1 and u_2 of Problem 4.19. Denote

$$
u(t,x) = u_1(t,x) - u_2(t,x).
$$

Then the function u is a solution of the problem

$$
\begin{cases}
\mathbb{D}^\gamma_{0+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = 0, \\
u(0,\lambda) = 0.
\end{cases}
$$

Then by applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23), we obtain

$$
\begin{cases}
\mathbb{D}^\gamma_{0+,t}\widehat{u}(t,\lambda) + \frac{m+\lambda^2}{1+a\lambda^2}\widehat{u}(t,\lambda) = 0, \\
\widehat{u}(0,\lambda) = 0.
\end{cases}
$$

Above equation has a trivial solution (see [48, p. 231, ex. 4.9]), i.e. $\widehat{u}(t,\lambda) = 0$ for all $(t,\lambda) \in Q_T$. Then using Theorem 2.44 (Plancherel Theorem) we have

$$
0 = \|\widehat{u}\|_{2,\alpha} = \|u\|_{2,\alpha}
$$

and $u(t,x) = 0$ for all $(t,x) \in Q_T$. Hence, uniqueness of the solution is proved. □

4.2.2. *Inverse source problem for the time-fractional pseudo-parabolic equation.* Now, let us study the inverse source problem.

Problem 4.22. *Let $0 < \gamma \leq 1$. Our aim to find a pair of functions (u, f) satisfying the equation*

$$\mathbb{D}^{\gamma}_{0+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), \quad (t,x) \in Q_T, \quad (4.34)$$

under the initial condition

$$u(0,x) = \phi(x), \quad x \in \mathbb{R} \quad (4.35)$$

and the over-determination condition

$$u(T,x) = \psi(x), \quad x \in \mathbb{R}, \quad (4.36)$$

where ϕ and ψ is sufficiently smooth functions.

Definition 4.23. A generalised solution of Problem 4.22 is a pair of functions (u, f), where

$$u \in C^{\gamma}([0,T], L^2(\mathbb{R}, d\mu_\alpha)) \cap C([0,T], \mathcal{H}_\alpha(\mathbb{R}, d\mu_\alpha)) \quad \text{and} \quad f \in L^2(\mathbb{R}, d\mu_\alpha)$$

satisfying the equation (4.34).

Theorem 4.24. *Let $0 < \gamma \leq 1$. We assume that $\psi, \phi \in \mathcal{H}_\alpha(\mathbb{R}, \mu_\alpha)$. Then generalised solution of Problem 4.22 exists, is unique, and can be written by the expressions*

$$f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} (m+\lambda^2)\frac{\psi(y) - \phi(y)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^{\gamma}\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^{\gamma}\right)} E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$

and

$$u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^{\gamma}\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^{\gamma}\right)}\psi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$
$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^{\gamma}\right) - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^{\gamma}\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^{\gamma}\right)}\phi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda).$$

Bewijs. We want to find a solution of Problem 4.22 by applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23) to the equation (4.34) and the conditions (4.35) and (4.36). Then it gives us

$$\mathbb{D}^{\gamma}_{0+,t}\widehat{u}(t,\lambda) + \frac{m+\lambda^2}{1+a\lambda^2}\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{1+a\lambda^2}, \quad (t,\lambda) \in Q_T, \quad (4.37)$$

and

$$\widehat{u}(0,\lambda) = \widehat{\phi}(\lambda), \quad \lambda \in \mathbb{R}, \quad (4.38)$$

$$\widehat{u}(T,\lambda) = \widehat{\psi}(\lambda), \quad \lambda \in \mathbb{R}, \quad (4.39)$$

where $\widehat{u}(t,\lambda)$ and $\widehat{f}(\lambda)$ are unknown. Let $0 < \gamma \leq 1$. Using expression (4.33) we can find solution of the equation (4.37) with initial condition (4.38), given by

$$\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{m+\lambda^2} + \left(\widehat{\phi}(\lambda) - \frac{\widehat{f}(\lambda)}{m+\lambda^2}\right)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^{\gamma}\right), \quad (4.40)$$

where $\widehat{f}(\lambda)$ is unknown and $\mathbb{E}_{\gamma,1}$ is the Mittag-Leffler function. Then applying the condition (4.39) to the expression (4.40) we obtain

$$\widehat{u}(T,\lambda) = \frac{\widehat{f}(\lambda)}{m+\lambda^2} + \left(\widehat{\phi}(\lambda) - \frac{\widehat{f}(\lambda)}{m+\lambda^2}\right)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right) = \widehat{\psi}(\lambda)$$

Thus, we can find unknown $\widehat{f}(\lambda)$ as following

$$\widehat{f}(\lambda) = (m+\lambda^2)\frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}. \tag{4.41}$$

Consequently, by substituting $\widehat{f}(\lambda)$ into (4.40), we have

$$\widehat{u}(t,\lambda) = \frac{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\widehat{\psi}(\lambda) + \frac{\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right) - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\widehat{\phi}(\lambda). \tag{4.42}$$

Then, we obtain the solution of Problem 4.22, which is a pair of functions (u, f) are given by the formulas

$$f(x) = \int_{\mathbb{R}}\int_{\mathbb{R}}(m+\lambda^2)\frac{\psi(y) - \phi(y)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$

and

$$u(t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\psi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda)$$
$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right) - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\phi(y)E_\alpha(x,\lambda)E_\alpha(-y,\lambda)d\mu_\alpha(y)d\mu_\alpha(\lambda).$$

by using the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ (2.24) for (4.41) and (4.42).

Let $\psi, \phi \in \mathcal{H}_\alpha(\mathbb{R}, \mu_\alpha)$. Then for f we have the following estimate

$$\begin{aligned}
\|f\|^2_{2,\alpha} = \|\widehat{f}\|^2_{2,\alpha} &= \int_{\mathbb{R}}|\widehat{f}(\lambda)|^2 d\mu_\alpha(\lambda)\\
&= \int_{\mathbb{R}}(m+\lambda^2)^2\left|\frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\right|^2 d\mu_\alpha(\lambda)\\
&\lesssim \int_{\mathbb{R}}(m+\lambda^2)^2|\widehat{\psi}(\lambda)|^2 d\mu_\alpha(\lambda) + \int_{\mathbb{R}}(m+\lambda^2)^2|\widehat{\phi}(\lambda)|^2 d\mu_\alpha(\lambda)\\
&\lesssim \|\psi\|^2_{\mathcal{H}_\alpha} + \|\phi\|^2_{\mathcal{H}_\alpha} < +\infty
\end{aligned}$$

Thus, we have $f \in L^2(\mathbb{R}, d\mu_\alpha)$. For $u(t,\cdot)$ we obtain

$$\|u(t,\cdot)\|^2_{\mathcal{H}_\alpha} = \int_{\mathbb{R}}(1+\lambda^2)^2|\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda)$$

$$
\begin{aligned}
&\lesssim \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} \widehat{\psi}(\lambda) \right|^2 d\mu_\alpha(\lambda) \\
&+ \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right) - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} \widehat{\phi}(\lambda) \right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\psi\|^2_{\mathcal{H}_\alpha} + \|\phi\|^2_{\mathcal{H}_\alpha}.
\end{aligned}
$$

Consequently, it gives

$$
\|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|\psi\|^2_{\mathcal{H}_\alpha} + \|\phi\|^2_{\mathcal{H}_\alpha} < +\infty.
$$

Rewriting the equation (4.37) as following

$$
\mathbb{D}^\gamma_{0+,t}\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{1+a\lambda^2} - \frac{m+\lambda^2}{1+a\lambda^2}\widehat{u}(t,\lambda),
$$

we have

$$
\begin{aligned}
\|\mathbb{D}^\gamma_{0+,t}u(t,\cdot)\|^2_{2,\alpha} &= \|\mathcal{F}_\alpha\left[\mathbb{D}^\gamma_{0+,t}u(t,\cdot)\right]\|^2_{2,\alpha} = \|\mathbb{D}^\gamma_{0+,t}\widehat{u}(t,\cdot)\|^2_{2,\alpha} \\
&= \int_{\mathbb{R}} \left| \frac{\widehat{f}(\lambda)}{1+a\lambda^2} - \frac{m+\lambda^2}{1+a\lambda^2}\widehat{u}(t,\lambda) \right|^2 d\mu_\alpha(\lambda) \lesssim \|f\|^2_{2,\alpha} + \|u(t,\cdot)\|^2_{\mathcal{H}_\alpha}.
\end{aligned}
$$

Thus,

$$
\|\mathbb{D}^\gamma_{0+,t}u\|^2_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} \lesssim \|f\|^2_{2,\alpha} + \|u\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} < +\infty.
$$

The existence is proved.

Now, we are going to prove uniqueness of the solution. Suppose that there are two solutions (u_1, f_1) and (u_2, f_2) of Problem 4.22. Denote

$$
u(t,x) = u_1(t,x) - u_2(t,x)
$$

and

$$
f(x) = f_1(x) - f_2(x).
$$

Then the functions u and f satisfy

$$
\begin{cases}
\mathbb{D}^\gamma_{0+,t}\left(u(t,x) - aD^2_{\alpha,x}u(t,x)\right) - D^2_{\alpha,x}u(t,x) + mu(t,x) = f(x), \\
u(0,\lambda) = 0, \\
u(T,\lambda) = 0.
\end{cases}
$$

Then by applying the Dunkl transform $\mathcal{F}_\alpha$ (2.23), we obtain

$$
\begin{cases}
\mathbb{D}^\gamma_{0+,t}\widehat{u}(t,\lambda) + \frac{m+\lambda^2}{1+a\lambda^2}\widehat{u}(t,\lambda) = \frac{\widehat{f}(\lambda)}{1+a\lambda^2}, \\
\widehat{u}(0,\lambda) = 0, \\
\widehat{u}(T,\lambda) = 0.
\end{cases}
$$

Via our calculation above, we can see that problem has a trivial solution, i.e. $\widehat{u}(t,\lambda) = 0$ and $\widehat{f}(\lambda) = 0$ for all $0 < t < T$, $\lambda \in \mathbb{R}$. Then using Theorem 2.44 (Plancherel Theorem) we able to see that $u(t,x) = 0$ and $f(x) = 0$ for all $0 < t < T$, $x \in \mathbb{R}$. Hence, uniqueness of the solution is proved. □

4.2.3. *Stability analysis for inverse source problem.* In this subsection we study stability of Problem 4.22.

Theorem 4.25. *Let (u, f) and (u_d, f_d) be solutions to Problem 4.22 corresponding to the data (ϕ, ψ) and its small perturbation (ϕ_d, ψ_d), respectively. Then the solution of Problem 4.22 depends continuously on these data, namely, we have*

$$\|u - u_d\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|\psi - \psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi - \phi_d\|^2_{\mathcal{H}_\alpha}$$

and

$$\|f - f_d\|^2_{2,\alpha} \lesssim \|\psi - \psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi - \phi_d\|^2_{\mathcal{H}_\alpha}.$$

Bewijs. From definition of the Dunkl transform

$$\mathcal{F}_\alpha[u(t,\cdot)](\lambda) = \widehat{u}(t,\lambda) = \int_{\mathbb{R}} u(t,x)E_\alpha(-x,\lambda)d\mu_\alpha(x)$$

we have

$$\begin{aligned}
\mathcal{F}_\alpha[u(t,\cdot) - u_d(t,\cdot)](\lambda) &= \int_{\mathbb{R}} (u(t,x) - u_d(t,x))E_\alpha(-x,\lambda)d\mu_\alpha(x) \\
&= \int_{\mathbb{R}} u(t,x)E_\alpha(-x,\lambda)d\mu_\alpha(x) - \int_{\mathbb{R}} u_d(t,x)E_\alpha(-x,\lambda)d\mu_\alpha(x) \\
&= \mathcal{F}_\alpha[u(t,\cdot)](\lambda) - \mathcal{F}_\alpha[u_d(t,\cdot)](\lambda) \\
&= \widehat{u}(t,\lambda) - \widehat{u}_d(t,\lambda),
\end{aligned}$$

here we have used property of the integral. Then we have

$$\|u(t,\cdot) - u_d(t,\cdot)\|^2_{\mathcal{H}_\alpha} = \int_{\mathbb{R}} (1+\lambda^2)^2 |\widehat{u}(t,\lambda) - \widehat{u}_d(t,\lambda)|^2 d\mu_\alpha(\lambda)$$

$$\begin{aligned}
&= \int_{\mathbb{R}} (1+\lambda^2)^2 \Bigg| \frac{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} \left(\widehat{\psi}(\lambda) - \widehat{\psi}_d(\lambda)\right) \\
&\qquad - \frac{\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right) - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} \left(\widehat{\phi}(\lambda) - \widehat{\phi}_d(\lambda)\right) \Bigg|^2 d\mu_\alpha(\lambda)
\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} \left(\widehat{\psi}(\lambda) - \widehat{\psi}_d(\lambda)\right) \right|^2 d\mu_\alpha(\lambda) \\
&\quad + \int_{\mathbb{R}} (1+\lambda^2)^2 \left| \frac{\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right) - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}t^\gamma\right)}{1 - \mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)} \left(\widehat{\phi}(\lambda) - \widehat{\phi}_d(\lambda)\right) \right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\psi - \psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi - \phi_d\|^2_{\mathcal{H}_\alpha}.
\end{aligned}$$

Consequently, we obtain

$$\|u - u_d\|^2_{C([0,T],\mathcal{H}_\alpha(\mathbb{R},d\mu_\alpha))} \lesssim \|\psi - \psi_d\|^2_{\mathcal{H}_\alpha} + \|\phi - \phi_d\|^2_{\mathcal{H}_\alpha}.$$

By writing (4.41) in a form

$$\widehat{f}(\lambda) = \frac{m+\lambda^2}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\widehat{\psi}(\lambda) - \frac{(m+\lambda^2)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m}{1+\lambda^2}T^\gamma\right)}\widehat{\phi}(\lambda)$$

we obtain

$$\begin{aligned}\|f-f_d\|_{2,\alpha}^2 = \|\widehat{f}-\widehat{f}_d\|_{2,\alpha}^2 &= \int_{\mathbb{R}} |\widehat{f}(\lambda)-\widehat{f}_d(\lambda)|^2 d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} \Bigg| \frac{m+\lambda^2}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\left(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\right) \\ &\quad - \frac{(m+\lambda^2)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\left(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\right)\Bigg|^2 d\mu_\alpha(\lambda)\end{aligned}$$

$$\begin{aligned}&\lesssim \int_{\mathbb{R}} \left| \frac{m+\lambda^2}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\left(\widehat{\psi}(\lambda)-\widehat{\psi}_d(\lambda)\right)\right|^2 d\mu_\alpha(\lambda) \\ &\quad + \int_{\mathbb{R}} \left| \frac{(m+\lambda^2)\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}{1-\mathbb{E}_{\gamma,1}\left(-\frac{m+\lambda^2}{1+a\lambda^2}T^\gamma\right)}\left(\widehat{\phi}(\lambda)-\widehat{\phi}_d(\lambda)\right)\right|^2 d\mu_\alpha(\lambda) \\ &\qquad \lesssim \|\psi-\psi_d\|_{\mathcal{H}_\alpha}^2 + \|\phi-\phi_d\|_{\mathcal{H}_\alpha}^2.\end{aligned}$$

Thus,

$$\|f-f_d\|_{2,\alpha}^2 \lesssim \|\psi-\psi_d\|_{\mathcal{H}_\alpha}^2 + \|\phi-\phi_d\|_{\mathcal{H}_\alpha}^2.$$

We completed our proofs. ∎

4.2.4. *Example for inverse source problem.* Here we test one sample case for the subject of the stability of the solution pair. Let us consider the following inverse source problem for the pseudo-parabolic equation

$$\frac{\partial}{\partial t}\left(u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x)\right) - \frac{\partial^2}{\partial x^2}u(t,x) + u(t,x) = f(x), \quad 0<t<1, \quad x\in\mathbb{R}$$

with Dirichlet boundary conditions

$$u(0,x) = u(1,x) = 0,$$

where $T=m=a=\gamma=1$, $\alpha=-\frac{1}{2}$, and $\phi(x)=\psi(x)=0$.

By applying Theorem 4.24, with $\gamma=1$ and $\alpha=-\frac{1}{2}$: our operator in time is $\frac{d}{dt}$ and in space $\Lambda^2_{-\frac{1}{2},x} = \frac{d^2}{dx^2}$, we obtain the trivial solution pair:

$$u(t,x)\equiv 0 \quad \text{and} \quad f(x)\equiv 0.$$

Now we consider a perturbation of the previous problem in the following form

$$\frac{\partial}{\partial t}\left(u_d(t,x) - \frac{\partial^2}{\partial x^2}u_d(t,x)\right) - \frac{\partial^2}{\partial x^2}u_d(t,x) + u_d(t,x) = f_d(x), \quad 0<t<1, \quad x\in\mathbb{R}$$

with conditions

$$u_d(0,x)=0, \quad \text{and} \quad u_d(1,x) = \epsilon\cdot\exp(-x^2), \quad \epsilon>0, \quad x\in\mathbb{R},$$

where $\phi_d(x) = 0$, $\psi_d(x) = \epsilon \cdot \exp(-x^2)$ ($\psi_d \in \mathcal{H}^2_\alpha(\mathbb{R}, \mu_\alpha)$). Then using Theorem 4.24, one obtains solution of the perturbation problem, expressed by

$$u_d(t,x) = \frac{\epsilon}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1-\exp(-t)}{1-\exp(-1)} \exp\left(-\frac{\lambda^2}{4}\right) \exp(ix\lambda)d\lambda \tag{4.43}$$

and

$$f_d(x) = \frac{\epsilon}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1+\lambda^2}{1-\exp(-1)} \exp\left(-\frac{\lambda^2}{4}\right) \exp(ix\lambda)d\lambda. \tag{4.44}$$

Integrals (4.43) and (4.44) are converges absolutely, because

$$\begin{aligned} |u_d(t,x)| &\leq \frac{\epsilon}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1-\exp(-t)}{1-\exp(-1)} \exp\left(-\frac{\lambda^2}{4}\right) d\lambda \\ &= \frac{\epsilon}{2\sqrt{\pi}} \frac{1-\exp(-t)}{1-\exp(-1)} \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda^2}{4}\right) d\lambda \leq \epsilon \frac{1-\exp(-T)}{1-\exp(-1)} \end{aligned}$$

and

$$|f_d(x)| \leq \frac{\epsilon}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1+\lambda^2}{1-\exp(-1)} \exp\left(-\frac{\lambda^2}{4}\right) d\lambda = \frac{3\epsilon}{1-\exp(-1)}.$$

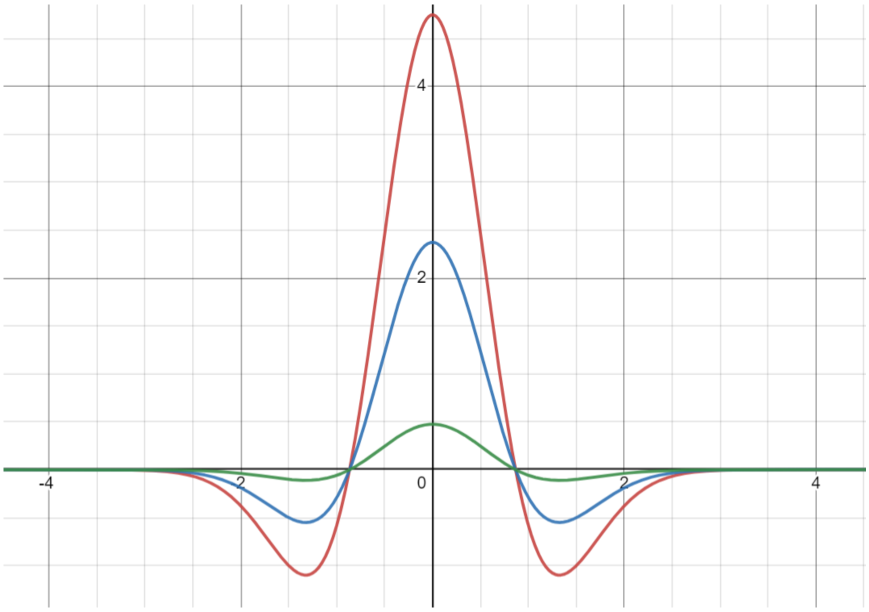
After a simple calculation, we see that the integrals (4.43) and (4.44) satisfy the equation and the conditions (perturbation problem). Indeed, integrals (4.43) and (4.44) can be represented as follows

$$\begin{aligned} u_d(t,x) &= \frac{\epsilon}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1-\exp(-t)}{1-\exp(-1)} \exp\left(-\frac{\lambda^2}{4}\right) \exp(ix\lambda)d\lambda \\ &= \frac{\epsilon}{2\sqrt{\pi}} \frac{1-\exp(-t)}{1-\exp(-1)} \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda^2}{4}\right) \exp(ix\lambda)d\lambda = \frac{1-\exp(-t)}{1-\exp(-1)} \epsilon \cdot \exp(-x^2) \end{aligned}$$

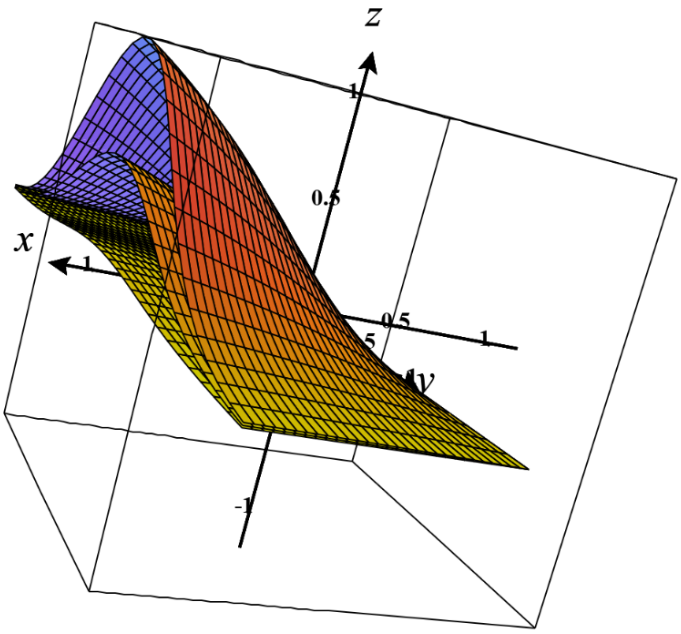
and

$$\begin{aligned} f_d(x) &= \frac{\epsilon}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1+\lambda^2}{1-\exp(-1)} \exp\left(-\frac{\lambda^2}{4}\right) \exp(ix\lambda)d\lambda \\ &= \frac{\epsilon}{2\sqrt{\pi}} \frac{1}{1-\exp(-1)} \int_{-\infty}^{\infty} (1+\lambda^2) \exp\left(-\frac{\lambda^2}{4}\right) \exp(ix\lambda)d\lambda \\ &= \frac{1}{1-\exp(-1)} \epsilon \cdot \exp(-x^2)(3-4x^2). \end{aligned}$$

In the following pictures, Figure 1 and Figure 2, you can find the graphics of the functions $f_d(x) = \frac{1}{1-\exp(-1)} \epsilon \cdot \exp(-x^2)(3-4x^2)$ and $u_d(x,y) = \frac{1-\exp(-x)}{1-\exp(-1)} \epsilon \cdot \exp(-y^2)$ for different epsilons ($\epsilon = 1, 0.5, 0.1$).



FIGUUR 1. The graph of the function f_d, here we have used desmos.com. The red graph with $\epsilon = 1$, the blue graph with $\epsilon = 0.5$, and the green graph with $\epsilon = 0.1$.



FIGUUR 2. The graph of the function u_d, here we have used 3D Calc Plotter. The upper graph with $\epsilon = 1$, the middle graph with $\epsilon = 0.5$, and the lower graph with $\epsilon = 0.1$.

Now, let us calculate the following integrals:

$$\|\psi-\psi_d\|_{\mathcal{H}^2_\alpha}=\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|(1+\lambda^2)\frac{\epsilon}{\sqrt{2}}\exp\left(-\frac{\lambda^2}{4}\right)\right|^2 d\lambda\right)^{\frac{1}{2}}$$
$$=\epsilon\left(\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}}(1+\lambda^2)^2\exp\left(-\frac{\lambda^2}{2}\right)d\lambda\right)^{\frac{1}{2}}=\epsilon\sqrt{3},$$

since $\mathcal{F}(\psi-\psi_d)(\lambda)=-\frac{\epsilon}{\sqrt{2}}\exp\left(-\frac{\lambda^2}{4}\right)$,

$$\|f-f_d\|_{2,\alpha}=\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|f(x)-f_d(x)|^2\,dx\right)^{\frac{1}{2}}$$
$$=\epsilon\left(\frac{1}{\sqrt{2\pi}(1-\exp(-1))^2}\int_{\mathbb{R}}\left|\exp(-x^2)(3-4x^2)\right|^2dx\right)^{\frac{1}{2}}=\epsilon\frac{\sqrt{3}\exp(1)}{\exp(1)-1},$$

and

$$\|u-u_d\|_{C([0,1],\mathcal{H}^2_\alpha(\mathbb{R},\mu_\alpha))}=\max_{0\le t\le 1}\|u(t,\cdot)-u_d(t,\cdot)\|_{\mathcal{H}^2_\alpha}=\epsilon\sqrt{3}\max_{0\le t\le 1}\frac{1-\exp(-t)}{1-\exp(-1)}=\epsilon\sqrt{3},$$

since

$$\|u(t,\cdot)-u_d(t,\cdot)\|_{\mathcal{H}^2_\alpha}=\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+\lambda^2)^2\left|\widehat{u}(t,\lambda)-\widehat{u}_d(t,\lambda)\right|^2d\lambda\right)^{\frac{1}{2}}$$
$$=\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(1+\lambda^2)^2\left|\frac{\epsilon}{\sqrt{2}}\frac{1-\exp(-t)}{1-\exp(-1)}\exp\left(-\frac{\lambda^2}{4}\right)\right|^2d\lambda\right)^{\frac{1}{2}}$$
$$=\left(\frac{\epsilon^2}{2\sqrt{2\pi}}\left(\frac{1-\exp(-t)}{1-\exp(-1)}\right)^2\int_{\mathbb{R}}(1+\lambda^2)^2\exp\left(-\frac{\lambda^2}{2}\right)d\lambda\right)^{\frac{1}{2}}$$
$$=\epsilon\frac{1-\exp(-t)}{1-\exp(-1)}\left(\frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}}(1+\lambda^2)^2\exp\left(-\frac{\lambda^2}{2}\right)d\lambda\right)^{\frac{1}{2}}.$$

According to the above computations we able to build a according table for different values of epsilon ($\epsilon=1,0.5,0.1$).

ϵ	1	0.5	0.1
$\|\psi-\psi_d\|_{\mathcal{H}^2_\alpha}$	1.73205	0.86602	0.173205
$\|f-f_d\|_{2,\alpha}$	2.74006	1.37003	0.274006
$\|u-u_d\|_{C([0,1],\mathcal{H}^2_\alpha(\mathbb{R},\mu_\alpha))}$	1.73205	0.86602	0.173205

Tabel 1. Stability test

Conclusion. In this subsection, we have considered one inverse source problem and defined its solution using our calculus developed in this paper. Then to examine stability of its solution, we considered perturbation problem. Table 1 shows that solution of the inverse source problem is stable regarding to the small changes of the dates.

4.3. **Time-fractional heat equation with bi-ordinal Hilfer fractional derivative.** In this section, we study the inverse source problem for the heat equation

$$D_{0+,t}^{(\gamma_1,\gamma_2)s}u(t,x) = aD_{\alpha,x}^2 u(t,x) + p(t)f(x),$$

where $D_{0+}^{(\gamma_1,\gamma_2)s}$ $(0 < \gamma_1, \gamma_2 \leq 1,\ s \in [0,1])$ is the bi-ordinal Hilfer fractional derivative, D_α $(\alpha \geq -1/2)$ is the Dunkl operator, p is a given function and u and f are unknown functions, which we should define. The necessary information about the bi-ordinal Hilfer fractional derivative and the Dunkl operator can be found in the preliminaries.

Definition 4.26. Let us define the space $\mathcal{H}$, as following

$$\mathcal{H} := \{f \in L^2(\mathbb{R}, d\mu_\alpha) \quad : \quad D_\alpha^2 f \in L^2(\mathbb{R}, d\mu_\alpha)\}$$

and norm in this space is defined by

$$\|f\|_{\mathcal{H}} := \|f\|_{2,\alpha} + \|D_\alpha^2 f\|_{2,\alpha}.$$

Definition 4.27. We consider the spaces $C([0,T], L^2(\mathbb{R}, d\mu_\alpha))$ and $C([0,T], \mathcal{H})$, with the norms

$$\|f\|_{C([0,T],L^2(\mathbb{R},d\mu_\alpha))} := \max_{0\leq t\leq T} \|f(t,\cdot)\|_{2,\alpha}$$

and

$$\|f\|_{C([0,T],\mathcal{H})} := \max_{0\leq t\leq T} \|f(t,\cdot)\|_{\mathcal{H}},$$

respectively.

4.3.1. *Direct problem.* We consider the following Cauchy problem 4.30. The purpose of considering Problem 4.30 is to help ro solve the inverse problem because when dealing with the inverse problem, we need to know the unique solution of the direct problem.

Let us consider the equation

$$D_{0+,t}^{(\gamma_1,\gamma_2)s}u(t,x) = aD_{\alpha,x}^2 u(t,x) + f(t,x), \tag{4.45}$$

where $0 < \gamma_1, \gamma_2 \leq 1$, $s \in [0,1]$, $a > 0$ and $\alpha \geq -1/2$, on the domain Q_T.

Remark 4.28. In a case $s = 1$, the bi-ordinal Hilfer fractional derivative gives the Caputo fractional derivative, so results for this problem coincides with results of first section.

Definition 4.29. We will call the function u a regular solution if it satisfies regularity conditions

$$t^{1-\eta}u(\cdot,x) \in C[0,T], \text{ and } D_{0+,t}^{(\gamma_1,\gamma_2)s}u(\cdot,x), D_{\alpha,x}^2 u(\cdot,x) \in C(0,T),$$

and the equation (4.45) for all $(t,x) \in Q_T$, where $\eta := \gamma_2 + \mu(1-\gamma_2)$.

Problem 4.30. *Our aim is to find a regular solution u of the equation (4.45) on the domain Q_T, which satisfies the initial condition*

$$\lim_{t\to 0+} I_{0+,t}^{1-\eta}u(t,x) = \xi(x), \quad x \in \mathbb{R}, \tag{4.46}$$

where ξ is given continuous function.

Theorem 4.31. *We assume that $f \in C([0,T],\mathcal{H})$ and $I^{\delta}_{0+,t}\widehat{f}(t,\lambda)$ is finite for every fixed $\lambda \in \mathbb{R}$, $\xi \in \mathcal{H}$, and $\delta > 1/2$. Then Problem 4.30 has a unique solution $t^{1-\eta}u \in C([0,T],\mathcal{H})$ and $D^{(\gamma_1,\gamma_2)s}_{0+,t}u \in C([0,T],L^2(\mathbb{R},d\mu_\alpha))$. Moreover it has a expression*

$$u(t,x) = t^{\eta-1}\int_{\mathbb{R}} \widehat{\xi}(\lambda)\mathbb{E}_{\delta,\eta}(-a\lambda^2 t^{\delta})E_\alpha(x,\lambda)d\mu_\alpha(\lambda) + \int_{\mathbb{R}}\left[\int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^{\delta}\right]\widehat{f}(\tau,\lambda)d\tau\right]E_\alpha(x,\lambda)d\mu_\alpha(\lambda),$$

where $\delta := \gamma_2 + s(\gamma_1-\gamma_2)$.

Bewijs. **The existence of the solution**. We assume that $u(t,\cdot) \in L^2(\mathbb{R},d\mu_\alpha)$. Then we can interpret function $u(t,\cdot)$ as a tempered distribution and apply the Dunkl transform $\mathcal{F}_\alpha$ (2.23). Thus, we obtain ordinary differential equation

$$D^{(\gamma_1,\gamma_2)s}_{0+,t}\widehat{u}(t,\lambda) = -a\lambda^2\widehat{u}(t,\lambda) + \widehat{f}(t,\lambda), \quad \lambda \in \mathbb{R}, \ 0 < t < T < +\infty, \tag{4.47}$$

respect to the variable $t \in [0,T]$ with an initial condition

$$\lim_{t\to 0+} I^{(1-s)(1-\gamma_2)}_{0+,t}\widehat{u}(t,\lambda) = \widehat{\xi}(\lambda), \tag{4.48}$$

for every fixed $\lambda \in \mathbb{R}$. After using Remark 2.73, we are able to rewrite the equation (4.47) as

$$I^{\eta-\delta}_{0+}D^{\eta}_{0+}\widehat{u}(t,\lambda) = -a\lambda^2\widehat{u}(t,\lambda) + \widehat{f}(t,\lambda),$$

where $\eta := \gamma_2 + s(1-\gamma_2)$ and $\delta := \gamma_2 + s(\gamma_1-\gamma_2)$. Then applying the operator I^{δ}_{0+} we obtain

$$I^{\delta}_{0+}I^{\eta-\delta}_{0+}D^{\eta}_{0+}\widehat{u}(t,\lambda) = -a\lambda^2 I^{\delta}_{0+}\widehat{u}(t,\lambda) + I^{\delta}_{0+}\widehat{f}(t,\lambda)$$

or

$$I^{\eta}_{0+}D^{\eta}_{0+}\widehat{u}(t,\lambda) = -a\lambda^2 I^{\delta}_{0+}\widehat{u}(t,\lambda) + I^{\delta}_{0+}\widehat{f}(t,\lambda).$$

Here we suppose that $\widehat{u}(\cdot,\lambda) \in L^1(0,T)$. Therefore, we have

$$\widehat{u}(t,\lambda) - \frac{t^{\eta-1}}{\Gamma(\eta)}\left[\lim_{t\to 0+} I^{1-\eta}_{0+}\widehat{u}(t,\lambda)\right] = -a\lambda^2 I^{\delta}_{0+}\widehat{u}(t,\lambda) + I^{\delta}_{0+}\widehat{f}(t,\lambda)$$

or

$$\widehat{u}(t,\lambda) = \frac{\widehat{\xi}(\lambda)}{\Gamma(\eta)}t^{\eta-1} - \frac{a\lambda^2}{\Gamma(\delta)}\int_0^t \frac{\widehat{u}(\tau,\lambda)d\tau}{(t-\tau)^{1-\delta}} + \frac{1}{\Gamma(\delta)}\int_0^t \frac{\widehat{f}(\tau,\lambda)d\tau}{(t-\tau)^{1-\delta}}. \tag{4.49}$$

For every fixed $\lambda \in \mathbb{R}$ the equation (4.49) is the linear Volterra integral equation of the second kind. Solution of the Volterra equation can be found by using the method of successive approximations [48, p. 222-223] and it has a form

$$\widehat{u}(t,\lambda) = \widehat{\xi}(\lambda)t^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2 t^{\delta}) + \int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^{\delta}\right]\widehat{f}(\tau,\lambda)d\tau. \tag{4.50}$$

The function $\widehat{u}$, expressed by (4.50), is a solution of Problem (4.47) - (4.48) for every $\lambda \in \mathbb{R}$. In this stage let us check some statements about $\widehat{u}$ which we assumed before. So, we do following computations

$$\int_0^T |\widehat{u}(t,\lambda)|\,dt \le C_1\left|\widehat{\xi}(\lambda)\right|\int_0^T \frac{t^{\eta-1}}{1+a\lambda^2 t^{\delta}}dt + C_2\int_0^T\int_0^t \frac{(t-\tau)^{\delta-1}\left|\widehat{f}(\tau,\lambda)\right|}{1+a\lambda^2(t-\tau)^{\delta}}d\tau dt$$

$$\leq C_1\left|\widehat{\xi}(\lambda)\right|\int_0^T t^{\eta-1}dt + C_2\int_0^T\int_0^t \frac{\left|\widehat{f}(\tau,\lambda)\right|}{(t-\tau)^{1-\delta}}d\tau dt$$
$$= \frac{C_1T^\eta}{\eta}\left|\widehat{\xi}(\lambda)\right| + C_2\Gamma(\delta)\int_0^T I^\delta_{0+,t}\left|\widehat{f}(t,\lambda)\right|dt$$

and

$$I^{1-\eta}_{0+}\widehat{u}(t,\lambda) = \widehat{\xi}(\lambda)I^{1-\eta}_{0+}\left(t^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)\right)$$
$$+\int_0^t I^{1-\eta}_{0+}\left((t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]\right)\widehat{f}(\tau,\lambda)d\tau$$
$$= \widehat{\xi}(\lambda)\mathbb{E}_{\delta,1}(-a\lambda^2t^\delta) + \int_0^t (t-\tau)^{\delta-\eta}\mathbb{E}_{\delta,1+\delta-\eta}\left[-a\lambda^2(t-\tau)^\delta\right]\widehat{f}(\tau,\lambda)d\tau.$$

Then we have $\lim_{t\to 0+} I^{1-\eta}_{0+}\widehat{u}(t,\lambda) = \xi(\lambda)$. Now, applying the inverse Dunkl transform $\mathcal{F}^{-1}_\alpha$ (2.24) to the (4.50) we obtain solution of Problem 4.30, which has a form

$$u(t,x) = t^{\eta-1}\int_{\mathbb{R}}\widehat{\xi}(\lambda)\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)E_\alpha(x,\lambda)d\mu_\alpha(\lambda)$$
$$+\int_{\mathbb{R}}\left[\int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]\widehat{f}(\tau,\lambda)d\tau\right]E_\alpha(x,\lambda)d\mu_\alpha(\lambda). \quad (4.51)$$

In the beginning of our proof we made the assumption that $u(t,\cdot)\in L^2(\mathbb{R},d\mu_\alpha)$. Now let us proof that it is correct, indeed

$$\|u(t,\cdot)\|^2_{2,\alpha} = \|\widehat{u}(t,\cdot)\|^2_{2,\alpha}$$
$$= \int_{\mathbb{R}}|\widehat{u}(t,\lambda)|^2 d\mu_\alpha(\lambda)$$
$$\lesssim t^{2(\eta-1)}\int_{\mathbb{R}}\left|\widehat{\xi}(\lambda)\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)\right|^2 d\mu_\alpha(\lambda)$$
$$+\int_{\mathbb{R}}\left|\left[\int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]\widehat{f}(\tau,\lambda)d\tau\right]\right|^2 d\mu_\alpha(\lambda)$$
$$\lesssim \int_{\mathbb{R}}\frac{t^{2(\eta-1)}}{(1+a\lambda^2t^\delta)^2}\left|\widehat{\xi}(\lambda)\right|^2 d\mu_\alpha(\lambda)$$
$$+\int_{\mathbb{R}}\left(\int_0^t \frac{(t-\tau)^{\delta-1}}{1+a\lambda^2(t-\tau)^\delta}\left|\widehat{f}(\tau,\lambda)\right|d\tau\right)^2 d\mu_\alpha(\lambda).$$

Then using Hölder's inequality and positivity of the expressions $a\lambda^2t^\delta$ and $a\lambda^2(t-\tau)^\delta$, we obtain

$$\|u(t,\cdot)\|^2_{2,\alpha} \lesssim t^{2(\eta-1)}\int_{\mathbb{R}}\left|\widehat{\xi}(\lambda)\right|^2 d\mu_\alpha(\lambda)$$
$$+\left(\int_0^t (t-\tau)^{2(\delta-1)}d\tau\right)\int_{\mathbb{R}}\left(\int_0^t\left|\widehat{f}(\tau,\lambda)\right|^2 d\tau\right)d\mu_\alpha(\lambda).$$

After applying Fubini's theorem and supposing $\delta > 1/2$, we have

$$\|u(t,\cdot)\|_{2,\alpha}^2 \lesssim t^{2(\eta-1)} \int_{\mathbb{R}} \left|\widehat{\xi}(\lambda)\right|^2 d\mu_\alpha(\lambda) + t^{2\delta-1} \int_0^t \int_{\mathbb{R}} \left|\widehat{f}(\tau,\lambda)\right|^2 d\mu_\alpha(\lambda) d\tau.$$

Hence, $u(t,\cdot) \in L^2(\mathbb{R}, d\mu_\alpha)$, whenever $\xi, f(t,\cdot) \in L^2(\mathbb{R}, d\mu_\alpha)$. Now, using the same computations and assumptions as previously, we are able to calculate

$$\|D^2_{\alpha,x} u(t,\cdot)\|_{2,\alpha}^2 \lesssim t^{2(\eta-1)} \|D^2_{\alpha,x}\xi\|_{2,\alpha}^2 + t^{2\delta-1} \int_0^t \|D^2_{\alpha,x} f(t,\cdot)\|_{2,\alpha}^2 d\tau.$$

Hence, $u(t,\cdot) \in \mathcal{H}$, whenever $\xi, f(t,\cdot) \in \mathcal{H}$. After taking the maximum on variable $t \in [0,T]$, where $T < +\infty$, on both sides of the last inequality and using Fubini's theorem we are able to calculate

$$\max_{0\le t\le T} \|u(t,\cdot)\|_{2,\alpha}^2 \lesssim \|\xi\|_{2,\alpha}^2 + \max_{0\le t\le T} \|f(t,\cdot)\|_{2,\alpha}^2.$$

Finally, let us show that $D^{(\gamma_1,\gamma_2)s}_{0+,t} u(t,\cdot) \in L^2(\mathbb{R}, d\mu_\alpha)$. Hence, we need to calculate

$$\begin{aligned}\|D^{(\gamma_1,\gamma_2)s}_{0+,t} u(t,\cdot)\|_{2,\alpha}^2 &= \|\mathcal{F}_\alpha \left[D^{(\gamma_1,\gamma_2)s}_{0+,t} u(t,\cdot)\right] \|_{2,\alpha}^2 = \|D^{(\gamma_1,\gamma_2)s}_{0+,t} \widehat{u}(t,\cdot)\|_{2,\alpha}^2 \\ &= \|-a\lambda^2 \widehat{u}(t,\cdot) + \widehat{f}(t,\cdot)\|_{2,\alpha}^2 \lesssim a\|(-\lambda^2)\widehat{u}(t,\cdot)\|_{2,\alpha}^2 + \|\widehat{f}(t,\cdot)\|_{2,\alpha}^2 \\ &= a\|D^2_{\alpha,x} u(t,\cdot)\|_{2,\alpha}^2 + \|f(t,\cdot)\|_{2,\alpha}^2.\end{aligned}$$

Moreover, we have

$$\max_{0\le t\le T} \|D^{(\gamma_1,\gamma_2)s}_{0+,t} u(t,\cdot)\|_{2,\alpha}^2 \lesssim a \max_{0\le t\le T} \|D^2_{\alpha,x} u(t,\cdot)\|_{2,\alpha}^2 + \max_{0\le t\le T} \|f(t,\cdot)\|_{2,\alpha}^2.$$

Uniqueness of the direct problem. Let there be two solutions u_1 and u_2 of Problem 4.30. After we set $u = u_1 - u_2$. Then we obtain

$$\begin{cases} D^{(\gamma_1,\gamma_2)s}_{0+,t} u(t,x) = a D^2_{\alpha,x} u(t,x), \\ \lim_{t\to 0+} I^{1-\eta}_{0+} u(t,x) = 0. \end{cases}$$

Hence, Theorem 4.31 gives us unique solution of the above problem $u = u_1 - u_2 = 0$ for all $t \in [0,T]$ and $x \in \mathbb{R}$, which implies $u_1 = u_2$, thanks to Plancherel theorem 2.44. □

4.3.2. *Inverse problem.* In this subsection, we consider the main problem of our section, the inverse source problem generated by the bi-ordinal Hilfer operator $D^{(\gamma_1,\gamma_2)s}_{0+,t}$ $(0 < \gamma_1, \gamma_2 \le 1,\ s \in [0,1])$ and the Dunkl operator D^2_α $(\alpha \ge -1/2)$. We prove Theorem 4.34, where we show unique solvability of Problem 4.32.

Problem 4.32. *Let $0 < \gamma_1, \gamma_2 \le 1$, $s \in [0,1]$, $a > 0$ and $\alpha \ge -1/2$. Our aim is to find a solution pair (u, f) of the inverse source problem*

$$D^{(\gamma_1,\gamma_2)s}_{0+,t} u(t,x) = a D^2_{\alpha,x} u(t,x) + p(t) f(x), \tag{4.52}$$

on the domain Q_T, satisfying the conditions

$$\lim_{t\to 0+} I^{1-\eta}_{0+,t} u(t,x) = \phi(x), \quad x \in \mathbb{R}, \tag{4.53}$$

and

$$u(T,x) = \psi(x), \quad x \in \mathbb{R},$$

where the functions p, ϕ and ψ are given functions.

Remark 4.33. If $s=1$ and $p(t)=1$, then the results for this problem coincides with results of first section.

Theorem 4.34. *Let $\psi,\phi\in\mathcal{H}$. We assume that $p\in C[0,T]$ and*

$$C^* := \int_0^T (T-\tau)^{\delta-1}E_{\delta,\delta}\left[-a\lambda^2(T-\tau)^\delta\right]p(\tau)d\tau$$

is a finite well defined nonzero number for every $T>0$ and $\lambda\in\mathbb{R}$, and $\delta>1/2$. Then Problem 4.32 has a solution pair (u,f), where u is a regular solution, which are $f\in L^2(\mathbb{R},d\mu_\alpha)$ and $t^{1-\eta}u\in C([0,T],\mathcal{H})$ with $D_{0+,t}^{(\gamma_1,\gamma_2)s}u\in C([0,T],L^2(\mathbb{R},d\mu_\alpha))$, and expressed by

$$\begin{aligned} u(t,x) = t^{\eta-1}&\int_{\mathbb{R}}\widehat{\phi}(\lambda)\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)E_\alpha(x,\lambda)d\mu_\alpha(\lambda)\\ &+\int_{\mathbb{R}}\frac{\widehat{\psi}(\lambda)-\widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)}{C^*}\\ &\times\left(\int_0^t(t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]p(\tau)d\tau\right)E_\alpha(x,\lambda)d\mu_\alpha(\lambda)\end{aligned}$$

and

$$f(x)=\frac{1}{C^*}\int_{\mathbb{R}}\left(\widehat{\psi}(\lambda)-\widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)\right)E_\alpha(x,\lambda)d\mu_\alpha(\lambda).$$

Bewijs. **The existence of the solution**. As in previous section we suppose that $u(t,\cdot), f\in L^2(\mathbb{R},d\mu_\alpha)$. Then we can interpret functions $u(t,\cdot)$ and f as tempered distributions and apply the Dunkl transform $\mathcal{F}_\alpha$ (2.23). Hence, we obtain ordinary differential equation

$$D_{0+,t}^{(\gamma_1,\gamma_2)s}\widehat{u}(t,\lambda)=-a\lambda^2\widehat{u}(t,\lambda)+p(t)\widehat{f}(\lambda),\quad \lambda\in\mathbb{R},\ 0<t<T<+\infty, \tag{4.54}$$

respect to the variable $t\in[0,T]$ with the conditions

$$\lim_{t\to 0+} I_{0+,t}^{(1-s)(1-\gamma_2)}\widehat{u}(t,\lambda)=\widehat{\phi}(\lambda) \tag{4.55}$$

and

$$\widehat{u}(T,\lambda)=\widehat{\psi}(\lambda),$$

for every fixed $\lambda\in\mathbb{R}$. The analysis of previous Subsection 4.3.1 gives us the unique solution of the Cauchy problem (4.54) - (4.55) and it has a form

$$\begin{aligned}\widehat{u}(t,\lambda)=\widehat{\phi}(\lambda)t^{\eta-1}&\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)\\ &+\widehat{f}(\lambda)\int_0^t(t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]p(\tau)d\tau.\end{aligned} \tag{4.56}$$

Then applying the condition $\widehat{u}(T,\lambda)=\widehat{\psi}(\lambda)$ to the (4.56) we have

$$\widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)+\widehat{f}(\lambda)\int_0^T(T-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(T-\tau)^\delta\right]p(\tau)d\tau=\widehat{\psi}(\lambda).$$

From we define

$$\widehat{f}(\lambda) = \frac{1}{C^*}\left(\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)\right). \tag{4.57}$$

After substituting $\widehat{f}$ into (4.56) we obtain

$$\begin{aligned}\widehat{u}(t,\lambda) &= \widehat{\phi}(\lambda)t^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)\\ &\quad + \frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)}{C^*}\\ &\quad \times \int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]p(\tau)d\tau. \end{aligned} \tag{4.58}$$

After applying the inverse Dunkl transform $\mathcal{F}_\alpha^{-1}$ (2.24) to the (4.57) and (4.58) we obtain solution of Problem 4.32, which are expressed by

$$f(x) = \frac{1}{C^*}\int_{\mathbb{R}}\left(\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)\right)E_\alpha(x,\lambda)d\mu_\alpha(\lambda)$$

and

$$\begin{aligned}u(t,x) &= t^{\eta-1}\int_{\mathbb{R}}\widehat{\phi}(\lambda)\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)E_\alpha(x,\lambda)d\mu_\alpha(\lambda)\\ &\quad + \int_{\mathbb{R}}\frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)}{C^*}\\ &\quad \times\left(\int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right]p(\tau)d\tau\right)E_\alpha(x,\lambda)d\mu_\alpha(\lambda).\end{aligned}$$

Now let us show that $f \in L^2(\mathbb{R}, d\mu_\alpha)$, whenever $\psi, \phi \in L^2(\mathbb{R}, d\mu_\alpha)$. For this, we need to calculate

$$\begin{aligned}\|f\|_{2,\alpha}^2 &= \|\widehat{f}\|_{2,\alpha}^2\\ &= \frac{1}{|C^*|}\|\widehat{\psi} - \widehat{\phi}T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)\|_{2,\alpha}^2\\ &\lesssim \|\widehat{\psi}\|_{2,\alpha}^2 + T^{2(\eta-1)}\int_{\mathbb{R}}\left|\widehat{\phi}(\lambda)\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)\right|^2 d\mu_\alpha(\lambda)\\ &\lesssim \|\widehat{\psi}\|_{2,\alpha}^2 + T^{2(\eta-1)}\int_{\mathbb{R}}\frac{\left|\widehat{\phi}(\lambda)\right|^2}{(1+a\lambda^2T^\delta)^2}d\mu_\alpha(\lambda)\\ &\lesssim \|\widehat{\psi}\|_{2,\alpha}^2 + \|\widehat{\phi}\|_{2,\alpha}^2 < +\infty.\end{aligned}$$

Then the following computations

$$\begin{aligned}\|u(t,\cdot)\|_{2,\alpha}^2 &= \|\widehat{u}(t,\cdot)\|_{2,\alpha}^2\\ &\lesssim \int_{\mathbb{R}}\left|\widehat{\phi}(\lambda)t^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2t^\delta)\right|^2 d\mu_\alpha(\lambda)\\ &+ \int_{\mathbb{R}}\left|\frac{\widehat{\psi}(\lambda) - \widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)}{C^*}\right.\end{aligned}$$

$$\times \int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^{\delta}\right]p(\tau)d\tau\Big|^2 d\mu_\alpha(\lambda)$$

$$\lesssim t^{2(\eta-1)}\int_{\mathbb{R}}\frac{\left|\widehat{\phi}(\lambda)\right|^2}{(1+a\lambda^2t^{\delta})^2}d\mu_\alpha(\lambda)$$

$$+\frac{1}{|C^*|}\int_{\mathbb{R}}\left(\left|\widehat{\psi}(\lambda)-\widehat{\phi}(\lambda)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^{\delta})\right|\int_0^t\frac{(t-\tau)^{\delta-1}\,|p(\tau)|}{1+a\lambda^2(t-\tau)^{\delta}}d\tau\right)^2$$

$$\times\, d\mu_\alpha(\lambda)$$

and

$$\|u(t,\cdot)\|_{2,\alpha}^2 \lesssim t^{2(\eta-1)}\|\widehat{\phi}\|_{2,\alpha}^2 + \frac{\left(\Gamma(\delta)I_{0+}^{\delta}|p(t)|\right)^2}{|C^*|}\int_{\mathbb{R}}\left|\widehat{\psi}(\lambda)-\widehat{\phi}(\lambda)T^{\eta-1}E_{\delta,\eta}(-a\lambda^2T^{\delta})\right|^2 d\mu_\alpha(\lambda)$$

$$\lesssim t^{2(\eta-1)}\|\widehat{\phi}\|_{2,\alpha}^2+\left(I_{0+}^{\delta}|p(t)|\right)^2\|\widehat{\psi}\|_{2,\alpha}^2$$

gives us $u(t,\cdot)\in L^2(\mathbb{R},d\mu_\alpha)$, whenever $\psi,\phi\in L^2(\mathbb{R},d\mu_\alpha)$ and

$$\max_{0\le t\le T}\|u(t,\cdot)\|_{2,\alpha}^2\lesssim\|\widehat{\phi}\|_{2,\alpha}^2+\|\widehat{\psi}\|_{2,\alpha}^2.$$

Now, using previous computations we have

$$\|D_{\alpha,x}^2u(t,\cdot)\|_{2,\alpha}^2\lesssim t^{2(\eta-1)}\|D_{\alpha,x}^2\phi\|_{2,\alpha}^2+\left(I_{0+}^{\delta}|p(t)|\right)^2\|D_{\alpha,x}^2\psi\|_{2,\alpha}^2.$$

and

$$\max_{0\le t\le T}\|D_{\alpha,x}^2u(t,\cdot)\|_{2,\alpha}^2\lesssim\|D_{\alpha,x}^2\phi\|_{2,\alpha}^2+\|D_{\alpha,x}^2\psi\|_{2,\alpha}^2.$$

Thus, $u(t,\cdot)\in\mathcal{H}$, whenever $\phi,\psi\in\mathcal{H}$. Then we obtain

$$\|D_{0+,t}^{(\gamma_1,\gamma_2)s}u(t,\cdot)\|_{2,\alpha}^2=\|aD_{\alpha,x}^2u(t,\cdot)+p(t)f\|_{2,\alpha}^2\lesssim a\|D_{\alpha,x}^2u(t,\cdot)\|_{2,\alpha}^2+|p(t)|\|f\|_{2,\alpha}^2$$

and

$$\max_{0\le t\le T}\|D_{0+,t}^{(\gamma_1,\gamma_2)s}u(t,\cdot)\|_{2,\alpha}^2\lesssim a\cdot\max_{0\le t\le T}\|D_{\alpha,x}^2u(t,\cdot)\|_{2,\alpha}^2+\|f\|_{2,\alpha}^2\max_{0\le t\le T}|p(t)|.$$

The uniqueness of the solution. Let there are two solutions (u_1,f_1) and (u_2,f_2) of Problem 4.32. After we set $u=u_1-u_2$ and $f=f_1-f_2$. Then we obtain

$$\begin{cases}D_{0+,t}^{(\gamma_1,\gamma_2)s}u(t,x)=aD_{\alpha,x}^2u(t,x)+p(t)f(x),\\ \lim_{t\to0+}I_{0+,t}^{1-\eta}u(t,x)=0,\\ u(T,0)=0.\end{cases}$$

Hence, the Theorem 4.34 gives us unique solution of the above problem $u=u_1-u_2=0$ and $f=f_1-f_2=0$ for all $t\in[0,T]$ and $x\in\mathbb{R}$, which implies $u_1=u_2$ and $f_1=f_2$, thanks to Plancherel theorem 2.44. □

4.3.3. *Stability result.* In this subsection we show stability of Problem 4.32.

Theorem 4.35. *Let (u, f) and (u_p, f_p) be solutions to Problem 4.32 corresponding to the data (ϕ, ψ) and its small perturbation (ϕ_p, ψ_p), respectively. Then the solution of Problem 4.32 depends continuously on these data. Moreover, we obtain*

$$\|u - u_p\|^2_{C([0,T],\mathcal{H})} \lesssim \|\psi - \psi_p\|^2_{\mathcal{H}} + \|\phi - \phi_p\|^2_{\mathcal{H}}$$

and

$$\|f - f_p\|^2_{2,\alpha} \lesssim \|\psi - \psi_p\|^2_{2,\alpha} + \|\phi - \phi_p\|^2_{2,\alpha}.$$

Bewijs. Let (u, f) and (u_p, f_p) be solutions to Problem 4.32 corresponding to the data (ϕ, ψ) and (ϕ_p, ψ_p), respectively, which satisfy Theorem 4.34. Then we are able to calculate

$$\|u(t,\cdot) - u_p(t,\cdot)\|^2_{2,\alpha} = \|\mathcal{F}_\alpha[u(t,\cdot) - u_p(t,\cdot)]\|^2_{2,\alpha}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}} \left| \left(\widehat{\phi}(\lambda) - \widehat{\phi}_p(\lambda)\right) t^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2 t^\delta) \right|^2 d\mu_\alpha(\lambda) \\
&\quad + \int_{\mathbb{R}} \left| \frac{\left(\widehat{\psi}(\lambda) - \widehat{\psi_p}(\lambda)\right) - \left(\widehat{\phi}(\lambda) - \widehat{\phi}_p(\lambda)\right) T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2 T^\delta)}{C^*} \right. \\
&\qquad \left. \times \int_0^t (t-\tau)^{\delta-1}\mathbb{E}_{\delta,\delta}\left[-a\lambda^2(t-\tau)^\delta\right] p(\tau)d\tau \right|^2 d\mu_\alpha(\lambda)
\end{aligned}$$

$$\begin{aligned}
&\lesssim t^{2(\eta-1)} \int_{\mathbb{R}} \frac{\left|\widehat{\phi}(\lambda) - \widehat{\phi}_p(\lambda)\right|^2}{(1 + a\lambda^2 t^\delta)^2} d\mu_\alpha(\lambda) \\
&\quad + \frac{1}{|C^*|} \int_{\mathbb{R}} \left(\left| \left(\widehat{\psi}(\lambda) - \widehat{\psi_p}(\lambda)\right) - \left(\widehat{\phi}(\lambda) - \widehat{\phi}_p(\lambda)\right) T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2 T^\delta) \right| \right. \\
&\qquad \left. \times \int_0^t \frac{(t-\tau)^{\delta-1}|p(\tau)|}{1 + a\lambda^2(t-\tau)^\delta} d\tau \right)^2 d\mu_\alpha(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\|u(t,\cdot) - u_p(t,\cdot)\|^2_{2,\alpha} \lesssim t^{2(\eta-1)}\|\widehat{\phi} - \widehat{\phi}_p\|^2_{2,\alpha} \\
&\quad + \frac{\left(\Gamma(\delta) I^\delta_{0+}|p(t)|\right)^2}{|C^*|} \int_{\mathbb{R}} \left| \left(\widehat{\psi}(\lambda) - \widehat{\psi_p}(\lambda)\right) - \left(\widehat{\phi}(\lambda) - \widehat{\phi}_p(\lambda)\right) T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2 T^\delta) \right|^2 \\
&\qquad \times d\mu_\alpha(\lambda) \lesssim t^{2(\eta-1)}\|\widehat{\phi} - \widehat{\phi}_p\|^2_{2,\alpha} + \left(I^\delta_{0+}|p(t)|\right)^2 \|\widehat{\psi} - \widehat{\psi_p}\|^2_{2,\alpha}.
\end{aligned}$$

Then using previous computations we obtain

$$\begin{aligned}
&\|D^2_{\alpha,x}\left(u(t,\cdot) - u_p(t,\cdot)\right)\|^2_{2,\alpha} \\
&\qquad \lesssim t^{2(\eta-1)}\|D^2_{\alpha,x}\left(\phi - \phi_p\right)\|^2_{2,\alpha} + \left(I^\delta_{0+}|p(t)|\right)^2 \|D^2_{\alpha,x}\left(\psi - \psi_p\right)\|^2_{2,\alpha}
\end{aligned}$$

and

$$\max_{0\le t\le T} \|D^2_{\alpha,x}\left(u(t,\cdot) - u_p(t,\cdot)\right)\|^2_{2,\alpha} \lesssim \|D^2_{\alpha,x}\left(\phi - \phi_p\right)\|^2_{2,\alpha} + \|D^2_{\alpha,x}\left(\psi - \psi_p\right)\|^2_{2,\alpha}.$$

Also for f, we can obtain the same estimates

$$\begin{aligned}
\|f-f_p\|_{2,\alpha}^2 &= \|\mathcal{F}_\alpha[f-f_p]\|_{2,\alpha}^2 \\
&= \|\mathcal{F}_\alpha[f]-\mathcal{F}_\alpha[f_p]\|_{2,\alpha}^2 \\
&= \frac{1}{|C^*|}\|\left(\widehat{\psi}-\widehat{\psi_p}\right)-\left(\widehat{\phi}-\widehat{\phi}_p\right)T^{\eta-1}\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)\|_{2,\alpha}^2 \\
&\lesssim \|\widehat{\psi}-\widehat{\psi_p}\|_{2,\alpha}^2 + T^{2(\eta-1)}\int_{\mathbb{R}}\left|\left(\widehat{\phi}(\lambda)-\widehat{\phi}_p(\lambda)\right)\mathbb{E}_{\delta,\eta}(-a\lambda^2T^\delta)\right|^2 d\mu_\alpha(\lambda) \\
&\lesssim \|\widehat{\psi}-\widehat{\psi_p}\|_{2,\alpha}^2 + \|\widehat{\phi}-\widehat{\phi}_p\|_{2,\alpha}^2.
\end{aligned}$$

Hence, we have

$$\|f-f_p\|_{2,\alpha}^2 \lesssim \|\psi-\psi_p\|_{2,\alpha}^2 + \|\phi-\phi_p\|_{2,\alpha}^2.$$

□

Conclusion and Future Work Outlook

In this thesis, we develop analysis of pseudo-differential operators and considered some inverse source problems generated by the Dunkl operators on the real line. Let us review the obtained results in this thesis:

In Chapter 3, we considered pseudo-differential operators generated by the Dunkl operator. We proved that these operators are continuous linear operators on $\mathcal{S}(\mathbb{R})$. We defined amplitude, adjoint and transpose operators and proved that they are also continuous linear operators on $\mathcal{S}(\mathbb{R})$. Then we studied distributional and convolution kernels of the pseudo-differential operators generated by the Dunkl operator and proved certain properties of them. In the last section, we considered boundedness results of the pseudo-differential operators under some assumptions. The results of this chapter are unpublished, and we hope that some papers will follow after the PhD defense.

We can envision two possible continuations of the results from Chapter 3:

- Can we developer a symbolic calculus for this analysis? To answer this question, we may start from revising [37].
- Can we extend obtained results to the higher dimensions, especially in $\mathbb{R}^n$?

Also, we can try to find some application to this analysis.

In Chapter 4, we studied some inverse source problems generated by the Dunkl operator. More precisely, inverse source problems for heat and pseudo-parabolic equations with Caputo and bi-ordinal Hilfer fractional differential operators, generated by the Dunkl operator. We obtained well-posedness results in the sense of Hadamard. First two parts of this chapter is based on our published work [10] (joint work with D. Serikbaev and N. Tokmagambetov) and as a preprint in ärxivïn [9] in 2023 (joint work with N. Tokmagambetov).

All problems considered in Chapter 4 are linear and have constant coefficients. A possible continuation of the results in this chapter is to explore linear problems with variable coefficients and to extend the analysis to non-linear problems. Additionally, these considerations can be extended to higher dimensions, especially in $\mathbb{R}^n$. For this kind of problems, it may be useful to consider discrete Dunkl analysis, so questions raises in this direction as well.

The author of the thesis has published several papers during his PhD, including [9, 10] and [11, 12, 13, 14, 15, 16].

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