Lecture 3
Methods of solution of the finite difference equations. Parabolic equations-1

A typical parabolic equation is the unsteady diffusion problem characterized by

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \tag{4.2.1}$$

An explicit finite difference equation scheme for (4.2.1) may be written in the forward difference in time and central difference in space (FTCS) as (see Figure 4.2.1a)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{\Delta x^2} + O(\Delta t, \Delta x^2) \tag{4.2.2a}$$

or

$$u_i^{n+1} = u_i^n + d\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \tag{4.2.2b}$$

where d is the diffusion number

$$d = \frac{\alpha \Delta t}{\Delta x^2} \tag{4.2.3}$$

By definition, (4.2.2) is explicit because u_i^{n+1} at time step $n + 1$ can be solved *explicitly* in terms of the known quantities at the previous time step n, thus called an *explicit scheme*.

$$u_i^n = \bar{u}_i^n + \varepsilon_i^n \tag{4.2.4}$$

and substituting (4.2.4) into (4.2.2a), we obtain

$$\frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\Delta t} + \frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(\bar{u}_{i+1}^n - 2\bar{u}_i^n + \bar{u}_{i-1}^n\right) + \frac{\alpha}{(\Delta x)^2}\left(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n\right) \tag{4.2.5}$$

or

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2}\left(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n\right) \tag{4.2.6}$$

Any finite mesh function, such as ε_i^n or the full solution u_i^n, can be decomposed into a Fourier series

$$\varepsilon_i^n = \sum_{j=-N}^{N} \bar{\varepsilon}_j^n e^{I k_j (i \Delta x)} = \sum_{j=-N}^{N} \bar{\varepsilon}_j^n e^{I j i \pi / N} \tag{4.2.13}$$

with $I = \sqrt{-1}$, $\bar{\varepsilon}_j^n$ being the amplitude of the j^{th} harmonic, and the spatial phase angle ϕ is given as

$$\phi = k_j \Delta x = j\pi / N \tag{4.2.14}$$

with $\phi = \pi$ corresponding to the highest frequency resolvable on the mesh, namely the frequency of the wavelength $2\Delta x$. Thus

$$\varepsilon_i^n = \sum_{j=-N}^{N} \bar{\varepsilon}_j^n e^{I i \phi} \tag{4.2.15}$$

Substituting (4.2.15) into (4.2.6) yields

$$\frac{\bar{\varepsilon}^{n+1} - \bar{\varepsilon}^{n}}{\Delta t} e^{Ii\phi} = \frac{\alpha}{\Delta x^2} \left(\bar{\varepsilon}^n e^{I(i+1)\phi} - 2\bar{\varepsilon}^n e^{Ii\phi} + \bar{\varepsilon}^n e^{I(i-1)\phi} \right)$$

or

$$\bar{\varepsilon}^{n+1} - \bar{\varepsilon}^{n} - d\bar{\varepsilon}^n (e^{I\phi} - 2 + e^{-I\phi}) = 0 \tag{4.2.16}$$

The computational scheme is said to be stable if the amplitude of any error harmonic $\bar{\varepsilon}^n$ does not grow in time, that is, if the following ratio holds:

$$|g| = \left| \frac{\bar{\varepsilon}^{n+1}}{\bar{\varepsilon}^{n}} \right| \leq 1 \quad \text{for all } \phi \tag{4.2.17}$$

where $g = \bar{\varepsilon}^{n+1}/\bar{\varepsilon}^n$ is the amplification factor, and is a function of time step Δt, frequency, and the mesh size Δx. It follows from (4.2.16) that

$$g = 1 + d(e^{I\phi} - 2 + e^{-I\phi}) \tag{4.2.18a}$$

or

$$g = 1 - 2d(1 - \cos\phi) \tag{4.2.18b}$$

Thus, the stability condition is

$$g \leq 1 \tag{4.2.19}$$

or

$$1 - 2d(1 - \cos\phi) \geq -1 \tag{4.2.20}$$

Since the maximum of $1 - \cos\phi$ is 2, we arrive at, for stability,

$$0 \leq d \leq 1/2 \tag{4.2.21}$$

OTHER EXPLICIT SCHEMES

Richardson Method

If the diffusion equation (4.2.1) is modeled by the form

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{\Delta x^2}, \quad O(\Delta t^2, \Delta x^2) \tag{4.2.22}$$

This is known as the Richardson method and is unconditionally unstable.

Dufort-Frankel Method

The finite difference equation for this method is given by

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\alpha\left(u_{i+1}^n - 2\dfrac{u_i^{n+1} + u_i^{n-1}}{2} + u_{i-1}^n\right)}{\Delta x^2} \tag{4.2.23a}$$

or

$$(1 + 2d)u_i^{n+1} = (1 - 2d)u_i^{n-1} + 2d\left(u_{i+1}^n + u_{i-1}^n\right), \quad O(\Delta t^2, \Delta x^2, (\Delta t/\Delta x)^2) \tag{4.2.23b}$$

This scheme can be shown to be unconditionally stable by the von Neumann stability analysis.