Lecture 9
Finite-difference methods of solutions of the incompressible flows.

ARTIFICIAL COMPRESSIBILITY METHOD

The governing equations for incompressible viscous flows, known as the incompressible Navier-Stokes system of equations, are written in nondimensionalized form as

Continuity

$$v_{i,i} = 0 \tag{5.2.1}$$

Momentum

$$\frac{\partial v_i}{\partial t} + v_{i,j}v_j = -p_{,i} + \frac{1}{Re}v_{i,jj} \tag{5.2.2}$$

where the following nondimensional quantities are used:

$$v_i = \frac{v_i^*}{v_\infty}, \qquad x_i = \frac{x_i^*}{L}, \qquad p = \frac{p^*}{\rho v_\infty^2}, \qquad t = \frac{t^* v_\infty}{L}, \qquad Re = \frac{v_\infty L}{\nu}$$

with asterisks implying the physical variable and *Re* being the Reynolds number.

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + v_{i,i} = 0 \tag{5.2.3}$$

where $\tilde{\rho}$ is an artificial density, equated to the product of artificial compressibility factor β and pressure,

$$\tilde{\rho} = \beta^{-1} p \tag{5.2.4}$$

Here $\frac{\partial \tilde{\rho}}{\partial \tilde{t}} \to 0$ at the steady state and $\tilde{t}$ is a fictitious time.

With these definitions and combining (5.2.1–5.2.4), we may write the incompressible Navier-Stokes system of equations in the form

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{W}}{\partial x_i} = \frac{1}{Re} \frac{\partial}{\partial x_i}\left(\mathbf{B}_{ij} \frac{\partial \mathbf{W}}{\partial x_j}\right) \tag{5.2.5}$$

with

$$\mathbf{W} = \begin{bmatrix} p \\ \mathrm{v}_j \end{bmatrix}, \qquad \mathbf{A}_i = \frac{\partial \mathbf{D}_i}{\partial \mathbf{W}}, \qquad \mathbf{D}_i = \begin{bmatrix} \beta \mathrm{v}_i \\ \mathrm{v}_i \mathrm{v}_j + p\delta_{ij} \end{bmatrix}, \qquad \mathbf{B}_{ij} = \begin{bmatrix} 0 & \\ & \delta_{ij} \end{bmatrix}$$

$$\mathbf{A}_1 = \frac{\partial \mathbf{D}_1}{\partial \mathbf{W}} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ 1 & 2u & 0 & 0 \\ 0 & \mathrm{v} & u & 0 \\ 0 & w & 0 & u \end{bmatrix} \qquad \mathbf{A}_2 = \frac{\partial \mathbf{D}_2}{\partial \mathbf{W}} = \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & \mathrm{v} & u & 0 \\ 1 & 0 & 2\mathrm{v} & 0 \\ 0 & 0 & w & \mathrm{v} \end{bmatrix}$$

$$\mathbf{A}_3 = \frac{\partial \mathbf{D}_3}{\partial \mathbf{W}} = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & w & 0 & u \\ 0 & 0 & w & \mathrm{v} \\ 1 & 0 & 0 & 2w \end{bmatrix}$$

Let us now investigate the eigenvalues of $\mathbf{A}_i$,

$$|\mathbf{A}_i - \lambda_i \mathbf{I}| = 0$$

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} - \frac{1}{\mathrm{Re}} \mathbf{D} \nabla^2 \mathbf{q} = 0, \qquad p = a^2 \rho.$$

$$\mathbf{q} \equiv \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad \mathbf{F} \equiv \begin{bmatrix} a^2 u \\ u^2 + p \\ uv \end{bmatrix}, \quad \mathbf{G} \equiv \begin{bmatrix} a^2 v \\ uv \\ v^2 + p \end{bmatrix}, \quad \mathbf{D} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{A} \equiv \begin{bmatrix} 0 & a^2 & 0 \\ 1 & 2u & 0 \\ 0 & v & u \end{bmatrix}, \quad \mathbf{B} \equiv \begin{bmatrix} 0 & 0 & a^2 \\ 0 & v & u \\ 1 & 0 & 2v \end{bmatrix}. \qquad \mathbf{F} = \mathbf{A}\mathbf{q} - u\mathbf{D}\mathbf{q} \text{ и } \mathbf{G} = \mathbf{B}\mathbf{q} - v\,\mathbf{D}\mathbf{q}.$$

$$\frac{\Delta \mathbf{q}^{n+1}}{\Delta t} + 0.5 L_x (\mathbf{F}^n + \mathbf{F}^{n+1}) + 0.5 L_y (\mathbf{G}^n + \mathbf{G}^{n+1}) - \\ - \frac{0.5\mathbf{D}}{\mathrm{Re}} (L_{xx} + L_{yy}) (\mathbf{q}^n + \mathbf{q}^{n+1}) = 0,$$

$$\Delta \mathbf{q}^{n+1} = \mathbf{q}^{n+1} - \mathbf{q}^n$$

$$L_{xx} \mathbf{q}^n = (\mathbf{q}^n_{j-1,k} - 2\mathbf{q}^n_{j,\,k} + \mathbf{q}^n_{j+1,\,k}) / \Delta x^2.$$

$$\mathbf{F}^{n+1} \approx \mathbf{F}^n + \mathbf{A}^n \Delta \mathbf{q}^{n+1}, \quad \mathbf{G}^{n+1} \approx \mathbf{G}^n + \mathbf{B}^n \Delta \mathbf{q}^{n+1}, \quad \mathbf{q}^{n+1} \approx \mathbf{q}^n + \Delta \mathbf{q}^{n+1}.$$

$$\left\{ \mathbf{I} + 0.5\,\Delta t \left[L_x \mathbf{A}^n + L_y \mathbf{B}^n - \frac{\mathbf{D}}{\mathrm{Re}} (L_{xx} + L_{yy}) \right] \right\} \Delta \mathbf{q}^{n+1} = \mathrm{RHS},$$

$$\mathrm{RHS} = \Delta t \, [(\mathbf{D}/\mathrm{Re}) (L_{xx} + L_{yy})\, \mathbf{q}^n - L_x \mathbf{F} - L_y \mathbf{G}].$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) + \frac{\partial p}{\partial x} = \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right\},$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(uv) + \frac{\partial v^2}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right\}.$$

SEMI-IMPLICIT METHOD FOR PRESSURE-LINKED EQUATIONS (SIMPLE)

$$u_{j+1/2,\,k}\,\Delta y + v_{j,\,k+1/2}\,\Delta x - u_{j-1/2,\,k}\,\Delta y - v_{j,\,k-1/2}\,\Delta x = 0.$$

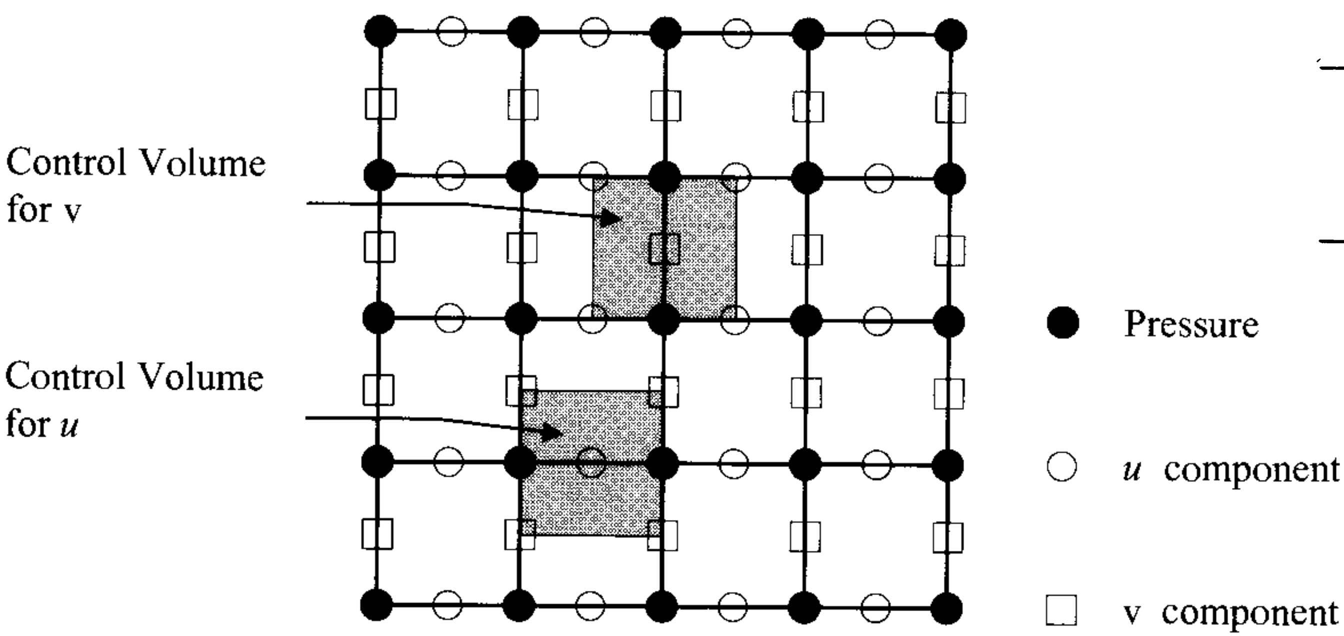
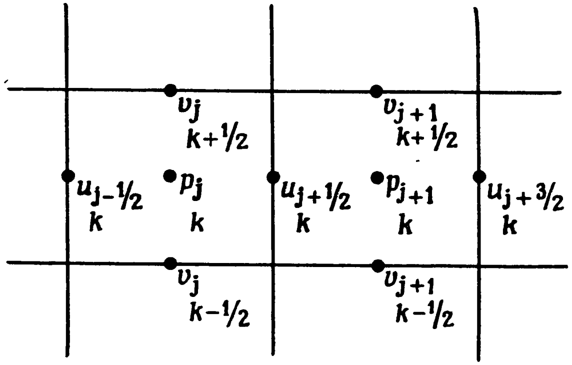


Figure 5.3.1 Computational domain for staggered grid.



in the algorithm known as SIMPLE [Patankar and Spalding, 1972]. In this method, the predictor-corrector procedure with successive pressure correction steps is used:

$$p = \overline{p} + p' \tag{5.3.1}$$

where p is the actual pressure, $\overline{p}$ is the estimated pressure, and p' is the pressure correction. Likewise, the actual velocity components in two-dimensions are

$$u = \overline{u} + u' \tag{5.3.2a}$$

$$\mathrm{v} = \overline{\mathrm{v}} + \mathrm{v}' \tag{5.3.2b}$$

The pressure corrections are related to the velocity corrections by approximate momentum equations,

$$\rho \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} \tag{5.3.3a}$$

$$\rho \frac{\partial \mathrm{v}'}{\partial t} = -\frac{\partial p'}{\partial y} \tag{5.3.3b}$$

or

$$u' = -\frac{\Delta t}{\rho} \frac{\partial p'}{\partial x} \tag{5.3.4a}$$

$$\mathrm{v}' = -\frac{\Delta t}{\rho} \frac{\partial p'}{\partial y} \tag{5.3.4b}$$

Combining (5.3.2) and (5.3.4) and substituting the result into the continuity equation, we obtain the so-called pressure-correction Poisson equation of an elliptic form,

$$p'_{,ii} = -\frac{\rho}{\Delta t}\left(\frac{\partial \mathrm{v}_i}{\partial x_i} - \frac{\partial \overline{\mathrm{v}}_i}{\partial x_i}\right) = \frac{\rho}{\Delta t}\frac{\partial \overline{\mathrm{v}}_i}{\partial x_i}, \qquad (i = 1,\ 2) \tag{5.3.5}$$

where we set $\frac{\rho}{\Delta t}\frac{\partial \mathrm{v}_i}{\partial x_i} = 0$ to enforce the mass conservation at the current iteration step. An iterative procedure is used to obtain a solution as follows [Raithby and Schneider, 1979].

(a) Guess the pressure $\overline{p}$ at each grid point.
(b) Solve the momentum equation to find $\overline{\mathrm{v}}_i$ at the staggered grid ($i + 1/2$, $i - 1/2$, $j + 1/2$, $j - 1/2$), discretized in control volumes and control surfaces (Section 1.4) as shown in Figure 5.3.1.
(c) Solve the pressure correction equation (5.3.5) to find p' at (i, j), $(i, j - 1)$, $(i, j + 1)$, $(i - 1, j)$, $(i + 1, j)$. Since the corner grid points are avoided, the scheme is "semi-implicit," not fully implicit, as shown in Figure 5.3.1.
(d) Correct the pressure and velocity using (2.2.9b), (5.3.2), and (5.3.4).

$$p = \overline{p} + p'$$

$$u = \overline{u} - \frac{\Delta t}{2\rho\,\Delta x}(p'_{i+1,j} - p'_{i-1,j}) - \frac{\Delta t}{\rho}\left(A^{(1)}_{i+\frac{1}{2},j} - A^{(1)}_{i-\frac{1}{2},j}\right) \tag{5.3.6}$$

$$\mathrm{v} = \overline{\mathrm{v}} - \frac{\Delta t}{2\rho\,\Delta y}(p'_{i,j+1} - p'_{i,j-1}) - \frac{\Delta t}{\rho}\left(A^{(2)}_{i,j+\frac{1}{2}} - A^{(2)}_{i,j-\frac{1}{2}}\right)$$

where

$$A^{(1)} = (\rho \mathrm{v}'_k \mathrm{v}'_1)_{,k} - \mu\left(\mathrm{v}'_{1,kk} + \frac{1}{3}\mathrm{v}'_{k,k1}\right) \quad (k = 1, 2)$$

$$A^{(2)} = (\rho \mathrm{v}'_k \mathrm{v}'_2)_{,k} - \mu\left(\mathrm{v}'_{2,kk} + \frac{1}{3}\mathrm{v}'_{k,k2}\right) \quad (k = 1, 2)$$

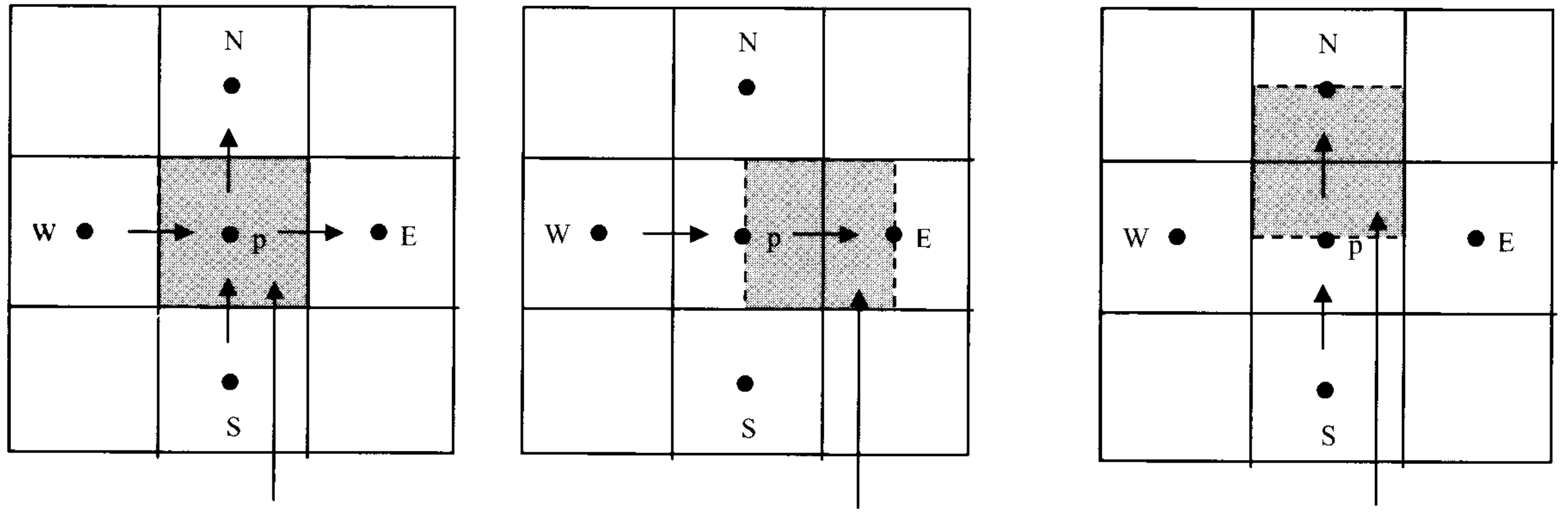
with μ being the dynamic viscosity.

(e) Replace the previous intermediate values of pressure and velocity $(\overline{p}, \overline{\mathrm{v}}_i)$ with the new corrector values (p, v_i) and return to (b).

(f) Repeat Steps (b) through (e) until convergence.

Often the convergence of the above process is not satisfactory because of the tendency for overestimation of p'. A remedy to this difficulty may be found by the use of under-relaxation parameter α,

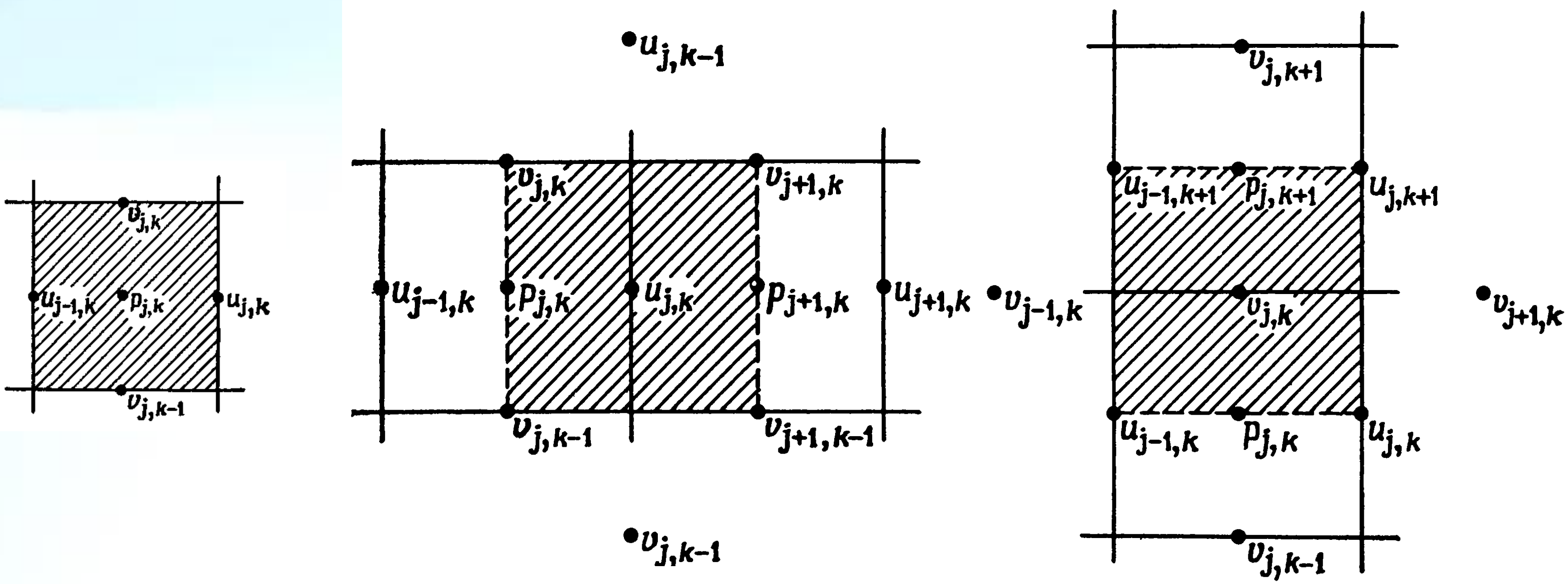
$$p = \overline{p} + \alpha p' \tag{5.3.7}$$



Control volume for p

Control volume for u

Control volume for v



PRESSURE IMPLICIT WITH SPLITTING OF OPERATORS

The governing equations consist of the momentum equation and pressure correction equation written as follows:

Momentum

$$\frac{\rho}{\Delta t}\left(\mathrm{v}_j^{n+1} - \mathrm{v}_j^n\right) = -s_{ij,i}^{n+1} - p_{,j}^{n+1} \tag{5.3.13}$$

Pressure Corrector

$$p_{,jj}^{n+1} = -\frac{\rho}{\Delta t}\left(\mathrm{v}_{j,j}^{n+1} - \mathrm{v}_{j,j}^n\right) - s_{ij,ij}^{n+1} \tag{5.3.14}$$

where $s_{ij,ij}$ refers to the derivatives of the sum of convection and viscous diffusion terms, $s_{ij,i}$.

$$s_{ij,i} = (\rho \mathrm{v}_i \mathrm{v}_j)_{,i} - \tau_{ij,i} \tag{5.3.15a}$$

$$\tau_{ij} = \mu(\mathrm{v}_{i,j} + \mathrm{v}_{j,i}) - \frac{2\mu}{3}\mathrm{v}_{k,k}\delta_{ij} \tag{5.3.15b}$$

(a) Predictor

$$\frac{\rho}{\Delta t}\left(v_j^* - v_j^n\right) = -s_{ij,i}^* - p_{,j}^n \qquad (5.3.16)$$

(b) Corrector I

$$p_{,jj}^* = -\frac{\rho}{\Delta t}\left(v_{j,j}^* - v_{j,j}^n\right) - s_{ij,ij}^* = \frac{\rho}{\Delta t} v_{j,j}^n - s_{ij,ij}^* \qquad (5.3.17)$$

$$\frac{\rho}{\Delta t}\left(v_j^{**} - v_j^n\right) = -s_{ij,i}^* - p_{,j}^* \qquad (5.3.18)$$

with $v_{j,j}^*$ set equal to zero in (5.3.17) in order to enforce the conservation of mass.

(c) Corrector II

$$p_{,jj}^{**} = \frac{\rho}{\Delta t} v_{j,j}^n - s_{ij,ij}^{**} \qquad (5.3.19)$$

$$\frac{\rho}{\Delta t}\left(v_j^{***} - v_j^n\right) = -s_{ij,i}^{**} - p_{,j}^{**} \qquad (5.3.20)$$

with $v_{j,j}^{**} = 0$ being once again enforced in (5.3.19). Thus, in the above process, there are no iterative steps involved.

MARKER-AND-CELL (MAC) METHOD

This is one of the earliest methods developed for the solution of incompressible flows, although its use in the original form is no longer pursued, but it has been altered to other more efficient schemes. The basic idea of MAC as originally introduced by Harlow and Welch [1965] is one of the pressure correction schemes developed on a staggered mesh, seeking to trace the paths of fictitious massless marker particles introduced on the free surface. The solution is advanced in time by solving the momentum equations for velocity components using the current estimates of the pressure distributions. The pressure is improved by numerically solving the Poisson equation,

$$p_{,ii} = f \tag{5.3.35}$$

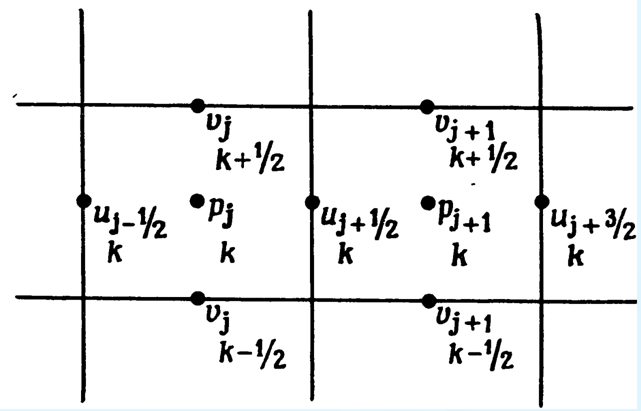
with

$$f = S - \frac{\partial D}{\partial t} \tag{5.3.36}$$

$$S = \lfloor -(\rho \mathrm{v}_i \mathrm{v}_j)_{,i} + \mu \mathrm{v}_{j,ii} \rfloor_{,j} \tag{5.3.37}$$

$$D = \mathrm{v}_{i,i} \tag{5.3.38}$$

Here, the correction in pressure is required to compensate for the nonzero dilatation D (5.3.38) at the current iteration level. The Poisson equation is then solved for the revised pressure field. The improved pressure may then be used in the momentum equations for a better solution at the present time step. If D does not vanish, cyclic process of solving the momentum equations and the Poisson equation is repeated until the velocity field is divergent free.



$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial}{\partial y}(uv)+\frac{\partial p}{\partial x}=\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right\},$$

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(uv)+\frac{\partial v^2}{\partial y}+\frac{\partial p}{\partial y}=\frac{1}{\mathrm{Re}}\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right\}.$$

$$\left[\frac{\partial u}{\partial t}\right]_{j+1/2,\,k}=\frac{\left(u^{n+1}_{j+1/2,\,k}-u^{n}_{j+1/2,\,k}\right)}{\Delta t}+O(\Delta t),$$

$$\left[\frac{\partial u^2}{\partial x}\right]_{j+1/2,\,k}=\frac{\left(u^2_{j+1,\,k}-u^2_{j,\,k}\right)}{\Delta x}+O(\Delta x^2),$$

$$\left[\frac{\partial (uv)}{\partial y}\right]_{j+1/2,\,k}=\frac{\left[(uv)_{j+1/2,\,k+1/2}-(uv)_{j+1/2,\,k-1/2}\right]}{\Delta y}+O(\Delta y^2),\tag{17.7}$$

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_{j+1/2,\,k}=\frac{\left(u_{j+3/2,\,k}-2u_{j+1/2,\,k}+u_{j-1/2,\,k}\right)}{\Delta x^2}+O(\Delta x^2),$$

$$\left[\frac{\partial^2 u}{\partial y^2}\right]_{j+1/2,\,k}=\frac{\left(u_{j+1/2,\,k-1}-2u_{j+1/2,\,k}+u_{j+1/2,\,k+1}\right)}{\Delta y^2}+O(\Delta y^2),$$

$$\left[\frac{\partial p}{\partial x}\right]_{j+1/2,\,k}=\frac{\left(p_{j+1,\,k}-p_{j,\,k}\right)}{\Delta x}+O(\Delta x^2).$$

$$u^{n+1}_{j+1/2,\,k} = F^n_{j+1/2,\,k} - \frac{\Delta t}{\Delta x}\left[p^{n+1}_{j+1,\,k} - p^{n+1}_{j,\,k}\right], \tag{17.8}$$

$$F^n_{j+1/2,\,k} = u^n_{j+1/2,\,k} + \Delta t\left[\frac{u_{j+3/2,\,k} - 2u_{j+1/2,\,k} + u_{j-1/2,\,k}}{\mathrm{Re}\,\Delta x^2} + \right.$$
$$+ \frac{\{u_{j+1/2,\,k-1} - 2u_{j+1/2,\,k} + u_{j+1/2,\,k+1}\}}{\mathrm{Re}\,\Delta y^2} - \frac{\{u^2_{j+1,\,k} - u^2_{j,\,k}\}}{\Delta x} -$$
$$\left. - \frac{\{(uv)_{j+1/2,\,k+1/2} - (uv)_{j+1/2,\,k-1/2}\}}{\Delta y}\right]^n. \tag{17.9}$$

$$v^{n+1}_{j,\,k+1/2} = G^n_{j,\,k+1/2} - \frac{\Delta t}{\Delta y}\left[p^{n+1}_{j,\,k+1} - p^{n+1}_{j,\,k}\right], \tag{17.10}$$

$$G^n_{j,\,k+1/2} = v^n_{j,\,k+1/2} + \Delta t\left[\frac{\{v_{j+1,\,k+1/2} - 2v_{j,\,k+1/2} + v_{j-1,\,k+1/2}\}}{\mathrm{Re}\,\Delta x^2} + \right.$$
$$+ \frac{\{v_{j,\,k+3/2} - 2v_{j,\,k+1/2} + v_{j,\,k-1/2}\}}{\mathrm{Re}\,\Delta y^2} -$$
$$\left. - \frac{\{(uv)_{j+1/2,\,k+1/2} - (uv)_{j-1/2,\,k+1/2}\}}{\Delta x} - \frac{(v^2_{j,\,k+1} - v^2_{j,\,k})}{\Delta y}\right]^n. \tag{17.11}$$

$$D^{n+1}_{j,k} = \frac{\left(u^{n+1}_{j+1/2,\,k} - u^{n+1}_{j-1/2,\,k}\right)}{\Delta x} + \frac{\left(v^{n+1}_{j,\,k+1/2} - v^{n+1}_{j,\,k-1/2}\right)}{\Delta y} = 0,$$

$$\left[\frac{(p_{j-1,\,k} - 2p_{j,\,k} + p_{j+1,\,k})}{\Delta x^2} + \frac{(p_{j,\,k-1} - 2p_{j,\,k} + p_{j,\,k+1})}{\Delta y^2}\right]^{n+1} =$$

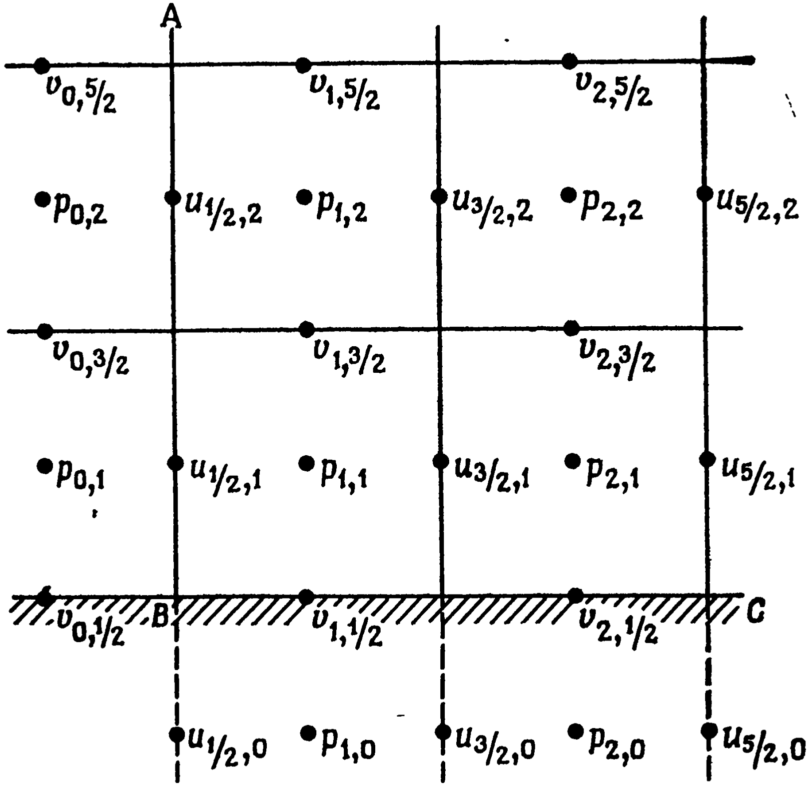
$$= \frac{1}{\Delta t}\left[\frac{\{F^n_{j+1/2,\,k} - F^n_{j-1/2,\,k}\}}{\Delta x} + \frac{\{G^n_{j,\,k+1/2} - G^n_{j,\,k-1/2}\}}{\Delta y}\right]. \qquad (17.13)$$

$$\mathrm{RHS}_{(17.13)} = \frac{D^n_{j,\,k}}{\Delta t} - [L_{xx}u^2_{j,\,k} + 2L_{xy}(uv)_{j,\,k} + L_{yy}v^2_{j,\,k} -$$
$$- (1/\mathrm{Re})\{L_{xx} + L_{yy}\}D_{j,\,k}]^n, \qquad (17.14)$$

где

$$L_{xx}u^2_{j,\,k} = (u^2_{j-1,\,k} - 2u^2_{j,\,k} + u^2_{j+1,\,k})/\Delta x^2,$$

$$L_{xy}(uv)_{j,\,k} = \{(uv)_{j+1/2,\,k+1/2} - (uv)_{j-1/2,\,k+1/2} - (uv)_{j+1/2,\,k-1/2} +$$
$$+ (uv)_{j-1/2,\,k-1/2}\}/\Delta x\,\Delta y.$$



$$v_{1,\,1/2} = v_{2,\,1/2} = \ldots = 0,$$
$$v_{0,\,3/2} = 2v_{1/2,\,3/2} - u_{1,\,3/2}.$$

$$\left.\frac{\partial u}{\partial x}\right|_{AB} = 0, \qquad \left.\frac{\partial v}{\partial x}\right|_{AB} = 0.$$

$$u_{3/2,\,2} = u_{-1/2,\,2}.$$

$$\frac{p_{2,\,1} - p_{2,\,0}}{\Delta y} = \frac{1}{\mathrm{Re}}\,\frac{v_{2,\,3/2} - 2v_{2,\,1/2} + v_{2,\,-1/2}}{\Delta y^2}$$

$$\partial p/\partial y = (\partial^2 v/\partial y^2)/\mathrm{Re},$$

$$u_{3/2,\,1/2} = 0 = 0.5\,(u_{3/2,\,1} + u_{3/2,\,0}) \quad \text{или} \quad u_{3/2,\,0} = -u_{3/2,\,1}.$$

$$v_{2,\,-1/2} = v_{2,\,3/2}, \qquad p_{2,\,0} = p_{2,\,1} - \frac{2v_{2,\,3/2}}{\mathrm{Re}\,\Delta y}. \qquad \partial v/\partial y = 0.$$