Lecture 7
Navier-Stokes system of equations. Conservative form.

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{F}}{\partial y}+\frac{\partial \mathbf{G}}{\partial z}=0,$$

$$\mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ E_t\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ (E_t+p)u\end{bmatrix},$$

$$\mathbf{F}=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ (E_t+p)v\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ (E_t+p)w\end{bmatrix}.$$

$$\rho=\rho(T,\ p).$$

$$p=(\gamma-1)\rho e,\quad T=\frac{(\gamma-1)e}{R}.$$

$$p=\rho RT,$$

$$e=c_vT,\quad h=c_pT,\quad \gamma=\frac{c_p}{c_v},\quad c_v=\frac{R}{\gamma-1},\quad c_p=\frac{\gamma R}{\gamma-1},$$

$$a=\sqrt{\frac{dp}{d\rho}}=\sqrt{\frac{\gamma p}{\rho}}=\sqrt{\gamma RT}.$$

$$\mu=C_1\frac{T^{3/2}}{T+C_2},$$

$$\mathrm{Pr}=c_p\mu/k.$$

$$k=(c_p/\mathrm{Pr})\,\mu,$$

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{F}}{\partial y}+\frac{\partial \mathbf{G}}{\partial z}=0,$$

$$\mathbf{U}=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ E_t\end{bmatrix},\quad \mathbf{E}=\begin{bmatrix}\rho u\\ \rho u^2+p-\tau_{xx}\\ \rho uv-\tau_{xy}\\ \rho uw-\tau_{xz}\\ (E_t+p)u-u\tau_{xx}-v\tau_{xy}-w\tau_{xz}+q_x\end{bmatrix},$$

$$\mathbf{F}=\begin{bmatrix}\rho v\\ \rho uv-\tau_{xy}\\ \rho v^2+p-\tau_{yy}\\ \rho vw-\tau_{yz}\\ (E_t+p)v-u\tau_{xy}-v\tau_{yy}-w\tau_{yz}+q_y\end{bmatrix}, \qquad (5.44)$$

$$\mathbf{G}=\begin{bmatrix}\rho w\\ \rho uw-\tau_{xz}\\ \rho vw-\tau_{yz}\\ \rho w^2+p-\tau_{zz}\\ (E_t+p)w-u\tau_{xz}-v\tau_{yz}-w\tau_{zz}+q_z\end{bmatrix}.$$

$$\tau_{xx} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right), \quad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx},$$

$$\tau_{yy} = \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right), \quad \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \tau_{zx},$$

$$\tau_{zz} = \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right), \quad \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{zy}.$$

$$p = \rho R T, \quad e = c_v T, \quad h = c_p T, \quad \gamma = \frac{c_p}{c_v}, \quad c_v = \frac{R}{\gamma - 1}, \quad c_p = \frac{\gamma R}{\gamma - 1},$$

nondimensional quantities

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad t^* = \frac{t}{L/V_\infty},$$

$$u^* = \frac{u}{V_\infty}, \quad v^* = \frac{v}{V_\infty}, \quad w^* = \frac{w}{V_\infty}, \quad \mu^* = \frac{\mu}{\mu_\infty}, \quad \mathrm{Re}_L = \frac{\rho_\infty V_\infty L}{\mu_\infty}.$$

$$\rho^* = \frac{\rho}{\rho_\infty}, \quad p^* = \frac{p}{\rho_\infty V_\infty^2}, \quad T^* = \frac{T}{T_\infty}, \quad e^* = \frac{e}{V_\infty^2},$$

$$\frac{\partial \mathbf{U}^*}{\partial t^*} + \frac{\partial \mathbf{E}^*}{\partial x^*} + \frac{\partial \mathbf{F}^*}{\partial y^*} + \frac{\partial \mathbf{G}^*}{\partial z^*} = 0,$$

$$\mathbf{U}^* = \begin{bmatrix} \rho^* \\ \rho^* u^* \\ \rho^* v^* \\ \rho^* w^* \\ E_t^* \end{bmatrix},$$

$$\mathbf{E}^* = \begin{bmatrix} \rho^* u^* \\ \rho^* u^{*2} + p^* - \tau_{xx}^* \\ \rho^* u^* v^* - \tau_{xy}^* \\ \rho^* u^* w^* - \tau_{xz}^* \\ (E_t^* + p^*)\, u^* - u^* \tau_{xx}^* - v^* \tau_{xy}^* - w^* \tau_{xz}^* + q_x^* \end{bmatrix},$$

$$\mathbf{F}^* = \begin{bmatrix} \rho^* v^* \\ \rho^* u^* v^* - \tau_{xy}^* \\ \rho^* v^{*2} + p^* - \tau_{yy}^* \\ \rho^* v^* w^* - \tau_{yz}^* \\ (E_t^* + p^*)\, v^* - u^* \tau_{xy}^* - v^* \tau_{yy}^* - w^* \tau_{yz}^* + q_y^* \end{bmatrix},$$

$$\mathbf{G}^* = \begin{bmatrix} \rho^* w^* \\ \rho^* u^* w^* - \tau_{xz}^* \\ \rho^* v^* w^* - \tau_{yz}^* \\ \rho^* w^{*2} + p^* - \tau_{zz}^* \\ (E_t^* + p^*)\, w^* - u^* \tau_{xz}^* - v^* \tau_{yz}^* - w^* \tau_{zz}^* + q_z^* \end{bmatrix},$$

$$E_t^* = \rho^* \left(e^* + \frac{u^{*2} + v^{*2} + w^{*2}}{2} \right).$$

$$\tau_{xx}^* = \frac{2\mu^*}{3\,\mathrm{Re}_L} \left(2\frac{\partial u^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*} - \frac{\partial w^*}{\partial z} \right),$$

$$\tau_{yy}^* = \frac{2\mu^*}{3\,\mathrm{Re}_L} \left(2\frac{\partial v^*}{\partial y^*} - \frac{\partial u^*}{\partial x^*} - \frac{\partial w^*}{\partial z^*} \right),$$

$$\tau_{zz}^* = \frac{2\mu^*}{3\,\mathrm{Re}_L} \left(2\frac{\partial w^*}{\partial z^*} - \frac{\partial u^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*} \right),$$

$$\tau_{xy}^* = \frac{\mu^*}{\mathrm{Re}_L} \left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*} \right),$$

$$\tau_{xz}^* = \frac{\mu^*}{\mathrm{Re}_L} \left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*} \right),$$

$$\tau_{yz}^* = \frac{\mu^*}{\mathrm{Re}_L} \left(\frac{\partial v^*}{\partial z^*} + \frac{\partial w^*}{\partial y^*} \right),$$

$$q_x^* = \frac{\mu^*}{(\gamma - 1) M_\infty^2 \mathrm{Re}_L \mathrm{Pr}} \frac{\partial T^*}{\partial x^*},$$

$$q_y^* = \frac{\mu^*}{(\gamma - 1) M_\infty^2 \mathrm{Re}_L \mathrm{Pr}} \frac{\partial T^*}{\partial y^*}, \qquad M_\infty = V_\infty / \sqrt{\gamma R T_\infty},$$

$$q_z^* = \frac{\mu^*}{(\gamma - 1) M_\infty^2 \mathrm{Re}_L \mathrm{Pr}} \frac{\partial T^*}{\partial z^*}, \qquad p^* = (\gamma - 1) \rho^* e^*, \quad T^* = \frac{\gamma M_\infty^2 p^*}{\rho^*}.$$

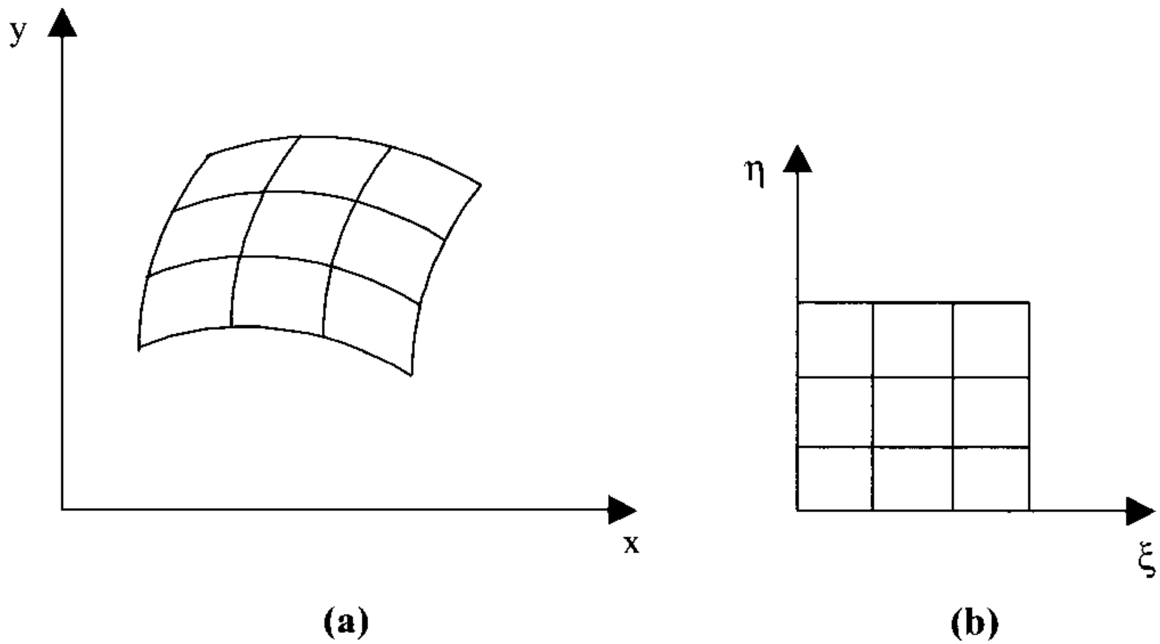
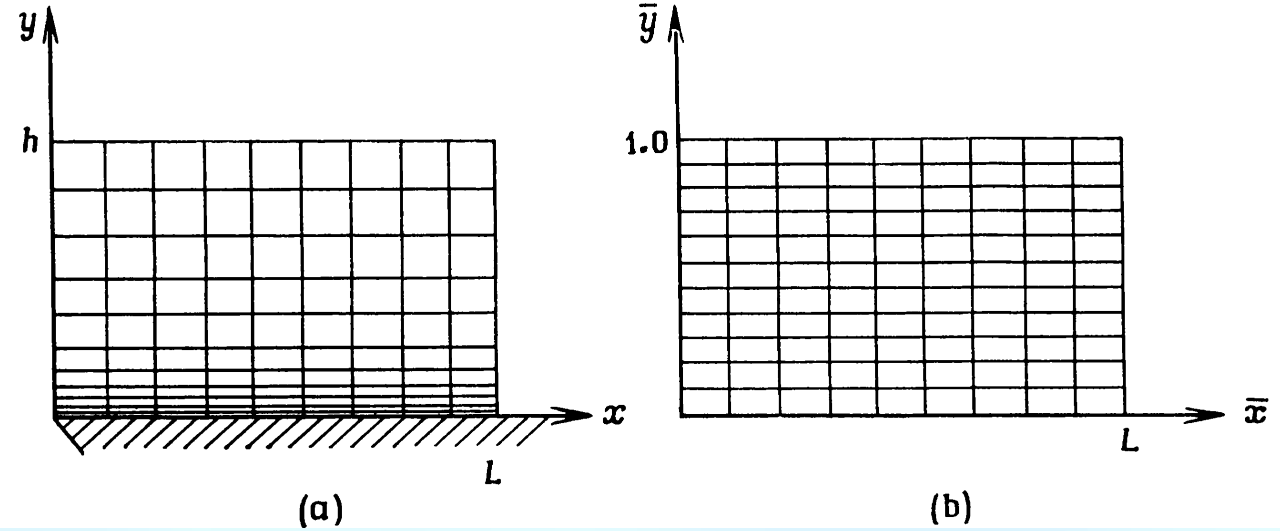


Figure 4.6.1 Transformation from curvilinear grid system into rectangular grid system. **(a)** Original curvilinear grid. **(b)** Transformed cartesian grid.



$$\bar{x} = x,$$

$$\bar{y} = 1 - \frac{\ln\{[\beta + 1 - (y/h)]/[\beta - 1 + (y/h)]\}}{\ln[(\beta + 1)/(\beta - 1)]}, \quad 1 < \beta < \infty.$$

$$\frac{\partial \bar{x}}{\partial y} = 0, \quad \frac{\partial \bar{y}}{\partial y} = \frac{2\beta}{h\{\beta^2 - [1 - (y/h)]^2\} \ln[(\beta + 1)/(\beta - 1)]}.$$

$$x = \bar{x},$$

$$y = h \frac{(\beta + 1) - (\beta - 1)\{[(\beta + 1)/(\beta - 1)]^{1-\bar{y}}\}}{[(\beta + 1)/(\beta - 1)]^{1-\bar{y}} + 1}.$$

$$
\begin{aligned}
\xi &= \xi(x,\ y,\ z),\\
\eta &= \eta(x,\ y,\ z),\\
\zeta &= \zeta(x,\ y,\ z),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} + \zeta_x \frac{\partial}{\partial \zeta},\\
\frac{\partial}{\partial y} &= \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} + \zeta_y \frac{\partial}{\partial \zeta},\\
\frac{\partial}{\partial z} &= \xi_z \frac{\partial}{\partial \xi} + \eta_z \frac{\partial}{\partial \eta} + \zeta_z \frac{\partial}{\partial \zeta}.
\end{aligned}
$$

$$
\begin{aligned}
d\xi &= \xi_x\, dx + \xi_y\, dy + \xi_z\, dz,\\
d\eta &= \eta_x\, dx + \eta_y\, dy + \eta_z\, dz,\\
d\zeta &= \zeta_x\, dx + \zeta_y\, dy + \zeta_z\, dz,
\end{aligned}
$$

$$
\begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.
$$

$$
\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}.
$$

$$
\begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & y_\zeta \end{bmatrix}^{-1} =
$$

$$
= J \begin{bmatrix} y_\eta z_\zeta - y_\zeta z_\eta & -(x_\eta z_\zeta - x_\zeta z_\eta) & x_\eta y_\zeta - x_\zeta y_\eta \\ -(y_\xi z_\zeta - y_\zeta z_\xi) & x_\xi z_\zeta - x_\zeta z_\xi & -(x_\xi y_\zeta - x_\zeta y_\xi) \\ y_\xi z_\eta - y_\eta z_\xi & -(x_\xi z_\eta - x_\eta z_\xi) & x_\xi y_\eta - x_\eta y_\xi \end{bmatrix}
$$

$$
\begin{aligned}
\xi_x &= J(y_\eta z_\zeta - y_\zeta z_\eta),\\
\xi_y &= -J(x_\eta z_\zeta - x_\zeta z_\eta),\\
\xi_z &= J(x_\eta y_\zeta - x_\zeta y_\eta),\\
\eta_x &= -J(y_\xi z_\zeta - y_\zeta z_\xi),\\
\eta_y &= J(x_\xi z_\zeta - x_\zeta z_\xi),\\
\eta_z &= -J(x_\xi y_\zeta - x_\zeta y_\xi),\\
\zeta_x &= J(y_\xi z_\eta - y_\eta z_\xi),\\
\zeta_y &= -J(x_\xi z_\eta - x_\eta z_\xi),\\
\zeta_z &= J(x_\xi y_\eta - x_\eta y_\xi),
\end{aligned}
$$

$$J = \frac{\partial(\xi, \eta. \zeta)}{\partial(x, y, z)} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix},$$

$$J = 1/J^{-1} = 1 \Big/ \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)} = 1 \Bigg/ \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} =$$

$$= 1/[x_\xi(y_\eta z_\zeta - y_\zeta z_\eta) - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi) + x_\zeta(y_\xi z_\eta - y_\eta z_\xi)].$$

$$\mathbf{U}_t + \xi_x \mathbf{E}_\xi + \eta_x \mathbf{E}_\eta + \zeta_x \mathbf{E}_\zeta + \xi_y \mathbf{F}_\xi + \eta_y \mathbf{F}_\eta + \zeta_y \mathbf{F}_\zeta + \\ + \xi_z \mathbf{G}_\xi + \eta_z \mathbf{G}_\eta + \zeta_z \mathbf{G}_\zeta = 0.$$

$$\left(\frac{\mathbf{U}}{J}\right)_t + \left(\frac{\mathbf{E}\xi_x + \mathbf{F}\xi_y + \mathbf{G}\xi_z}{J}\right)_\xi + \left(\frac{\mathbf{E}\eta_x + \mathbf{F}\eta_y + \mathbf{G}\eta_z}{J}\right)_\eta + \\ + \left(\frac{\mathbf{E}\zeta_x + \mathbf{F}\zeta_y + \mathbf{G}\zeta_z}{J}\right)_\zeta - \mathbf{E}\left[\left(\frac{\xi_x}{J}\right)_\xi + \left(\frac{\eta_x}{J}\right)_\eta + \left(\frac{\zeta_x}{J}\right)_\zeta\right] - \\ - \mathbf{F}\left[\left(\frac{\xi_y}{J}\right)_\xi + \left(\frac{\eta_y}{J}\right)_\eta + \left(\frac{\zeta_y}{J}\right)_\zeta\right] - \\ - \mathbf{G}\left[\left(\frac{\xi_z}{J}\right)_\xi + \left(\frac{\eta_z}{J}\right)_\eta + \left(\frac{\zeta_z}{J}\right)_\zeta\right] = 0. \qquad (5.238)$$

$$\mathbf{U}_1 = \frac{\mathbf{U}}{J},$$

$$\mathbf{E}_1 = \frac{1}{J}(\mathbf{E}\xi_x + \mathbf{F}\xi_y + \mathbf{G}\xi_z),$$

$$\mathbf{F}_1 = \frac{1}{J}(\mathbf{E}\eta_x + \mathbf{F}\eta_y + \mathbf{G}\eta_z),$$

$$\mathbf{G}_1 = \frac{1}{J}(\mathbf{E}\zeta_x + \mathbf{F}\zeta_y + \mathbf{G}\zeta_z)$$

$$\frac{\partial \mathbf{U}_1}{\partial t} + \frac{\partial \mathbf{E}_1}{\partial \xi} + \frac{\partial \mathbf{F}_1}{\partial \eta} + \frac{\partial \mathbf{G}_1}{\partial \zeta} = 0.$$

$$\tau_{xy} = \mu\left(\xi_y \frac{\partial u}{\partial \xi} + \eta_y \frac{\partial u}{\partial \eta} + \zeta_y \frac{\partial u}{\partial \zeta} + \xi_x \frac{\partial v}{\partial \xi} + \eta_x \frac{\partial v}{\partial \eta} + \zeta_x \frac{\partial v}{\partial \zeta}\right).$$