Lecture 11
Finite difference methods for compressible flows.
Euler equations.

Quasilinearization of Euler Equations

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}_i}{\partial x_i}=0, \quad \text{or} \quad \frac{\partial \mathbf{U}}{\partial t}+\mathbf{a}_i\frac{\partial \mathbf{U}}{\partial x_i}=0 \quad (i=1,2,3)$$

with

$$\mathbf{U}=\begin{bmatrix}\rho\\ \rho \mathrm{v}_j\\ \rho E\end{bmatrix}, \qquad \mathbf{F}_i=\begin{bmatrix}\rho \mathrm{v}_i\\ \rho \mathrm{v}_i \mathrm{v}_j+p\delta_{ij}\\ \rho E\mathrm{v}_i+p\mathrm{v}_i\end{bmatrix}, \qquad \mathbf{a}_i=\frac{\partial \mathbf{F}_i}{\partial \mathbf{U}},$$

$$\mathbf{a}_1=\begin{bmatrix}0 & 1 & 0 & 0\\ \frac{(\gamma-3)u^2}{2}+\frac{(\gamma-1)\mathrm{v}^2}{2} & (3-\gamma)u & -(\gamma-1)\mathrm{v} & \gamma-1\\ -u\mathrm{v} & \mathrm{v} & u & 0\\ -\gamma uE+(\gamma-1)uq^2 & \gamma E-\frac{\gamma-1}{2}(\mathrm{v}^2+3u^2) & -(\gamma-1)u\mathrm{v} & \gamma u\end{bmatrix}$$

(6.

$$\mathbf{a}_2=\begin{bmatrix}0 & 0 & 1 & 0\\ -u\mathrm{v} & \mathrm{v} & u & 0\\ \frac{(\gamma-3)\mathrm{v}^2}{2}+\frac{(\gamma-1)u^2}{2} & -(\gamma-1)u & (3-\gamma)\mathrm{v} & \gamma-1\\ -\gamma\mathrm{v}E+(\gamma-1)\mathrm{v}q^2 & -(\gamma-1)u\mathrm{v} & \gamma E-\frac{\gamma-1}{2}(u^2+3\mathrm{v}^2) & \gamma\mathrm{v}\end{bmatrix}$$

Alternatively, the Euler equations may be written in nonconservation form for isentropic flow in terms of the primitive variable $\mathbf{V}$ as

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{V}}{\partial x_i} = 0$$

$$\mathbf{V} = \begin{bmatrix} \rho \\ u \\ \mathrm{v} \\ p \end{bmatrix} = \begin{bmatrix} \rho \\ u \\ \mathrm{v} \\ (\gamma - 1)\left(\rho E - \rho \dfrac{(u^2 + \mathrm{v}^2)}{2}\right) \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & \dfrac{1}{\rho} \\ 0 & 0 & u & 0 \\ 0 & \rho a^2 & 0 & u \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} \mathrm{v} & 0 & \rho & 0 \\ 0 & \mathrm{v} & 0 & 0 \\ 0 & 0 & \mathrm{v} & \dfrac{1}{\rho} \\ 0 & 0 & \rho a^2 & \mathrm{v} \end{bmatrix}$$

Introducing a transformation between the conservation and nonconservation variables,

$$\mathbf{M} = \frac{\partial \mathbf{U}}{\partial \mathbf{V}}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{q^2}{2} & \rho u & \rho v & \frac{1}{\gamma - 1} \end{bmatrix} \quad \mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{-u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ \frac{-v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{\gamma - 1}{2} q^2 & -(\gamma - 1) u & -(\gamma - 1) v & \gamma - 1 \end{bmatrix}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{V}}{\partial x_i} = 0 \qquad \mathbf{A}_i = \mathbf{M}^{-1} \mathbf{a}_i \mathbf{M}, \qquad \mathbf{a}_i = \mathbf{M} \mathbf{A}_i \mathbf{M}^{-1}$$

$$|\mathbf{K} - \lambda \mathbf{I}| = 0 \qquad \mathbf{A}_i \kappa_i = \mathbf{K}$$

$$\mathbf{L}^{-1} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{L}^{-1} \mathbf{A} \frac{\partial \mathbf{V}}{\partial x} = 0, \quad \text{with} \quad \mathbf{L} = \frac{\partial \mathbf{V}}{\partial \mathbf{W}} \qquad \delta \mathbf{W} = \mathbf{L}^{-1} \delta \mathbf{V}$$

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{L}^{-1} \mathbf{A} \mathbf{L} \frac{\partial \mathbf{W}}{\partial x} = 0$$

$$\mathbf{P} = \frac{\partial \mathbf{U}}{\partial \mathbf{W}} = \frac{\partial \mathbf{U}}{\partial \mathbf{V}} \frac{\partial \mathbf{V}}{\partial \mathbf{W}} = \mathbf{M} \mathbf{L}, \qquad \mathbf{P}^{-1} = \mathbf{L}^{-1} \mathbf{M}^{-1}$$

$$\mathbf{P}^{-1}=\mathbf{L}^{-1}\mathbf{M}^{-1}=\begin{bmatrix} 1-\dfrac{\gamma-1}{2}\dfrac{u^2}{a^2} & (\gamma-1)\dfrac{u}{a} & \dfrac{-(\gamma-1)}{a^2} \\ \left(\dfrac{\gamma-1}{2}u^2-ua\right)\dfrac{1}{\rho a} & \dfrac{1}{\rho a}[a-(\gamma-1)u] & \dfrac{-(\gamma-1)}{\rho a} \\ -\left(\dfrac{\gamma-1}{2}u^2+ua\right)\dfrac{1}{\rho a} & \dfrac{1}{\rho a}[a+(\gamma-1)u] & \dfrac{-(\gamma-1)}{\rho a} \end{bmatrix}$$

$$\mathbf{P}=\mathbf{ML}=\begin{bmatrix} 1 & \dfrac{\rho}{2a} & -\dfrac{\rho}{2a} \\ u & \dfrac{\rho}{2a}(u+a) & -\dfrac{\rho}{2a}(u-a) \\ \dfrac{u^2}{2} & \dfrac{\rho}{2a}\left(\dfrac{u^2}{2}+ua+\dfrac{a^2}{\gamma-1}\right) & -\dfrac{\rho}{2a}\left(\dfrac{u^2}{2}-ua+\dfrac{a^2}{\gamma-1}\right) \end{bmatrix}$$

$$\mathbf{\Lambda}=\begin{bmatrix} u & & \\ & u+a & \\ & & u-a \end{bmatrix}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} + \begin{bmatrix} u & & \\ & u+a & \\ & & u-a \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = 0$$

$W_1 = \rho - \int \frac{dp}{a^2}$ = constant along the C_0 characteristic, stream line

$W_2 = u + \int \frac{dp}{\rho a}$ = constant along the C_+ characteristic

$W_3 = u - \int \frac{dp}{\rho a}$ = constant along the C_- characteristic

Lax-Friedrichs First Order Scheme

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} = 0$$

$$\mathbf{U}_{i,j}^{n+1} = \frac{1}{4}\left(\mathbf{U}_{i+1,j}^{n} + \mathbf{U}_{i-1,j}^{n} + \mathbf{U}_{i,j+1}^{n} + \mathbf{U}_{i,j-1}^{n}\right) - \frac{\tau_x}{2}\left(\mathbf{f}_{i+1,j}^{n} - \mathbf{f}_{i-1,j}^{n}\right) - \frac{\tau_y}{2}\left(\mathbf{g}_{i,j+1}^{n} - \mathbf{g}_{i,j-1}^{n}\right)$$

with

$$\tau_x = \frac{\Delta t}{\Delta x}, \quad \tau_y = \frac{\Delta t}{\Delta y}$$

and

$$\mathbf{U}_{i,j}^{n+1} = \frac{1}{4}\left(\mathbf{U}_{i+1,j}^{n} + \mathbf{U}_{i-1,j}^{n} + \mathbf{U}_{i,j+1}^{n} + \mathbf{U}_{i,j-1}^{n}\right)$$

It can be shown that the von Neumann analysis leads to the stability condition,

$$\tau_x^2(u+a)^2 + \tau_y^2(\mathrm{v}+a)^2 \le \frac{1}{2}$$

Lax-Wendroff Second Order Scheme

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^{n} - \tau_x \bar{\delta}_x \mathbf{f}_{i,j}^{n} - \tau_y \bar{\delta}_y \mathbf{g}_{i,j}^{n} + \frac{\tau_x^2}{2}\delta_x(\mathbf{a}_{i,j}\delta_x \mathbf{f}_{i,j}) + \frac{\tau_y^2}{2}\delta_y(\mathbf{b}_{i,j}\delta_y \mathbf{g}_{i,j})$$

$$+ \frac{\tau_x \tau_y}{2}\left[\bar{\delta}_x(\mathbf{a}_{i,j}\bar{\delta}_y \mathbf{g}_{i,j}) + \bar{\delta}_y(\mathbf{b}_{i,j}\bar{\delta}_x \mathbf{f}_{i,j})\right]$$

where

$$\bar{\delta}_x \mathbf{f}_{i,j} = \frac{1}{2}(\mathbf{f}_{i+1,j} - \mathbf{f}_{i-1,j}), \qquad \bar{\delta}_y \mathbf{g}_{i,j} = \frac{1}{2}(\mathbf{g}_{i,j+1} - \mathbf{g}_{i,j-1})$$

$$\delta_x \mathbf{f}_{i,j} = \mathbf{f}_{i+1,j} - \mathbf{f}_{i-1,j}, \qquad \delta_y \mathbf{g}_{i,j} = \mathbf{g}_{i,j+1} - \mathbf{g}_{i,j-1}$$

Step 1

$$\mathbf{U}_{i,j}^{n+\frac{1}{2}} = \frac{1}{4}\left(\mathbf{U}_{i+1,j}^{n} + \mathbf{U}_{i-1,j}^{n} + \mathbf{U}_{i,j+1}^{n} + \mathbf{U}_{i,j-1}^{n}\right) - \frac{\tau_x}{2}\left(\mathbf{f}_{i+1,j}^{n} - \mathbf{f}_{i-1,j}^{n}\right) - \frac{\tau_y}{2}\left(\mathbf{g}_{i,j+1}^{n} - \mathbf{g}_{i,j-1}^{n}\right)$$

Step 2

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^{n} - \tau_x\left(\mathbf{f}_{i+1,j}^{n+\frac{1}{2}} - \mathbf{f}_{i-1,j}^{n+\frac{1}{2}}\right) - \tau_y\left(\mathbf{g}_{i,j+1}^{n+\frac{1}{2}} - \mathbf{g}_{i,j-1}^{n+\frac{1}{2}}\right)$$

The stability condition is shown to be, for $\Delta x = \Delta y$

$$\frac{\Delta t}{\Delta x}(|\mathbf{v}| + a) \leq \frac{1}{\sqrt{2}}$$

CENTRAL SCHEMES WITH INDEPENDENT SPACE-TIME DISCRETIZATION

$$\frac{d\mathbf{U}_{i,j}}{dt} = -\frac{1}{2}\left[\frac{\mathbf{f}_{i+1,j} - \mathbf{f}_{i-1,j}}{\Delta x} + \frac{\mathbf{g}_{i,j+1} - \mathbf{g}_{i,j-1}}{\Delta y}\right]$$

where various finite difference schemes of the time derivative term may be applied. The two level time integration of (6.2.50) leads to

$$(1+\xi)\Delta\mathbf{U}^{n+1} - \xi\Delta\mathbf{U}^n = \Delta t\theta\left[\left(\frac{\partial\mathbf{f}}{\partial x} + \frac{\partial\mathbf{g}}{\partial y}\right)^{n+1} - \left(\frac{\partial\mathbf{f}}{\partial x} + \frac{\partial\mathbf{g}}{\partial y}\right)^{n}\right]$$

with $\xi > -1/2$, $\theta \geq 1/2(\xi + 1)$ for linear stability.

The two-level integration scheme takes the form

$$(1+\xi)\Delta\mathbf{U}^{n+1} + \Delta t\theta\left(\frac{\partial\mathbf{f}}{\partial x} + \frac{\partial\mathbf{g}}{\partial y}\right)^{n+1} = -\Delta t\left(\frac{\partial\mathbf{f}}{\partial x} + \frac{\partial\mathbf{g}}{\partial y}\right)^{n} + \xi\Delta\mathbf{U}^n$$

or

$$\left[(1+\xi) + \Delta t\theta\left(\frac{\partial\mathbf{a}}{\partial x} + \frac{\partial\mathbf{b}}{\partial y}\right)\right]\Delta\mathbf{U}^{n+1} = -\Delta t\left(\frac{\partial\mathbf{f}}{\partial x} + \frac{\partial\mathbf{g}}{\partial y}\right)^{n} + \xi\Delta\mathbf{U}^n$$

Introducing a central discretization, we obtain

$$[(1+\xi)+\theta(\tau_x\bar{\delta}_x\mathbf{a}+\tau_y\bar{\delta}_y\mathbf{b})]\Delta\mathbf{U}_{i,j}^{n+1} = -\tau_x\left(\bar{\delta}_x\mathbf{f}_{i,j}^n+\tau_y\bar{\delta}_y\mathbf{g}_{i,j}^n\right)+\xi\,\Delta\mathbf{U}_{i,j}^n \tag{6.2.55}$$

or

$$\begin{aligned}(1+\xi)\Delta\mathbf{U}_{i,j}^{n+1}&+\theta\frac{\Delta t}{2\Delta x}(\mathbf{a}_{i+1,j}\Delta\mathbf{U}_{i+1,j}-\mathbf{a}_{i-1,j}\Delta\mathbf{U}_{i-1,j})^{n+1}\\&+\theta\frac{\Delta t}{2\Delta y}(\mathbf{b}_{i,j+1}\Delta\mathbf{U}_{i,j+1}-\mathbf{b}_{i,j-1}\Delta\mathbf{U}_{i,j-1})^{n+1}\\&=-\Delta t\left(\frac{\mathbf{f}_{i+1,j}^n-\mathbf{f}_{i-1,j}^n}{2\Delta x}+\frac{\mathbf{g}_{i,j+1}^n-\mathbf{g}_{i,j-1}^n}{2\Delta y}\right)+\xi\,\Delta\mathbf{U}_{i,j}^n\end{aligned} \tag{6.2.56}$$

and with an ADI factorization, for $\xi=0$

$$(1+\theta\tau_x\bar{\delta}_x)(1+\theta\tau_y\bar{\delta}_y\mathbf{b}^n)\Delta\overline{\mathbf{U}}_{i,j} = -\left(\tau_x\bar{\delta}_x\mathbf{f}_{i,j}^n+\tau_y\bar{\delta}_y\mathbf{g}_{i,j}^n\right) \tag{6.2.57a}$$

$$(1+\theta\tau_y\bar{\delta}_y\mathbf{b}^n)\Delta\mathbf{U}_{i,j}^{n+1} = \Delta\overline{\mathbf{U}}_{i,j} \tag{6.2.57b}$$

Notice that each step is a tridiagonal system along the x lines for $\Delta\overline{\mathbf{U}}$ and along the y lines for $\Delta\mathbf{U}^{n+1}$.

FIRST ORDER UPWIND SCHEMES

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} + \frac{\partial \mathbf{g}}{\partial y} = 0$$

or

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} = 0$$

Flux Vector Splitting Method

$$\Lambda_1 = \mathbf{P}_1^{-1}\mathbf{A}\mathbf{P}_1 = \begin{bmatrix} u & & & \\ & u & & \\ & & u+a & \\ & & & u-a \end{bmatrix}$$

$$\Lambda_2 = \mathbf{P}_2^{-1}\mathbf{B}\mathbf{P}_2 = \begin{bmatrix} \mathrm{v} & & & \\ & \mathrm{v} & & \\ & & \mathrm{v}+a & \\ & & & \mathrm{v}-a \end{bmatrix}$$

$$\mathbf{f}^+ = \mathbf{f}, \quad \mathbf{f}^- = \mathbf{0} \quad \text{for supersonic flow}$$

$$\mathbf{f}^+ = \mathbf{0}, \quad \mathbf{f}^- = \mathbf{f} \quad \text{for subsonic flow}$$

and

$$\Lambda_1^+ = \begin{bmatrix} u & & & \\ & u & & \\ & & u+a & \\ & & & 0 \end{bmatrix}, \quad \Lambda_1^- = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & u-a \end{bmatrix}$$

with

$\mathbf{A}^+ = \mathbf{P}_1\Lambda_1^+\mathbf{P}_1^{-1}, \quad \mathbf{A}^- = \mathbf{P}_1\Lambda_1^-\mathbf{P}_1^{-1}$ and similarly for **B**. The split fluxes are defined by

$$\mathbf{f}^\pm = \mathbf{A}^\pm\mathbf{U} \quad \mathbf{g}^\pm = \mathbf{B}^\pm\mathbf{U}$$

which will allow the split flux components to be written as follows:

$$f = \frac{\rho}{2\gamma}\begin{bmatrix} \eta \\ \eta u + a(\lambda_2 - \lambda_3) \\ \eta v \\ \eta\frac{u^2+v^2}{2} + ua(\lambda_2 - \lambda_3) + a^2\frac{\lambda_2+\lambda_3}{\gamma-1} \end{bmatrix}$$

with

$$g = \frac{\rho}{2\gamma}\begin{bmatrix} \eta \\ \eta u \\ \eta v + a(\lambda_2 - \lambda_3) \\ \eta\frac{u^2+v^2}{2} + va(\lambda_2 - \lambda_3) + a^2\frac{\lambda_2+\lambda_3}{\gamma-1} \end{bmatrix} \qquad \eta = 2(\gamma-1)\lambda_1 + \lambda_2 + \lambda_3$$

Rewriting (6.2.58a) in a discrete form for a variable cross section

$$\mathbf{U}_{i.j}^{n+1} - \mathbf{U}_{i,j}^{n} = -\frac{\Delta t}{\Delta x}\left(\mathbf{f}^*_{i+\frac{1}{2},j} - \mathbf{f}^*_{i-\frac{1}{2},j}\right) - \frac{\Delta t}{\Delta y}\left(\mathbf{g}^*_{i,j+\frac{1}{2}} - \mathbf{g}^*_{i.j-\frac{1}{2}}\right)$$

with

$$\mathbf{f}^*_{i+\frac{1}{2},j} = \mathbf{f}^-_{i+1,j} + \mathbf{f}^+_{i.j}, \qquad \mathbf{g}^*_{i,j+\frac{1}{2}} = \mathbf{g}^-_{i,j+1} + \mathbf{g}^+_{i,j}$$

SECOND ORDER UPWIND SCHEMES (MUSCL)

$$\mathbf{f}^{**}_{i+\frac{1}{2}} = \frac{1}{2}\left[\mathbf{f}\left(\mathbf{U}^{L}_{i+\frac{1}{2}}\right) + \mathbf{f}\left(\mathbf{U}^{R}_{i+\frac{1}{2}}\right) - |\mathbf{a}|_{i+\frac{1}{2}}\left(\mathbf{U}^{R}_{1+\frac{1}{2}} - \mathbf{U}^{L}_{i+\frac{1}{2}}\right)\right]$$

with ** representing the second order scheme and

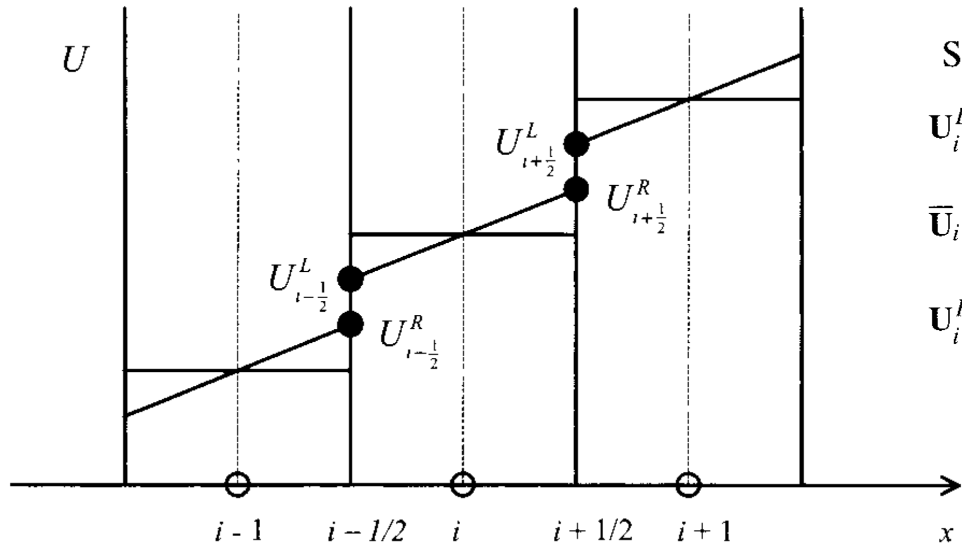
$$\mathbf{U}^{L}_{i+\frac{1}{2}} = \mathbf{U}_i + \frac{1}{4}[(1-\kappa)(\mathbf{U}_i - \mathbf{U}_{i-1}) + (1+\kappa)(\mathbf{U}_{i+1} - \mathbf{U}_i)]$$

$$\mathbf{U}^{R}_{i+\frac{1}{2}} = \mathbf{U}_{i+1} - \frac{1}{4}[(1+\kappa)(\mathbf{U}_{i+1} - \mathbf{U}_i) + (1-\kappa)(\mathbf{U}_{i+2} - \mathbf{U}_{i+1})]$$

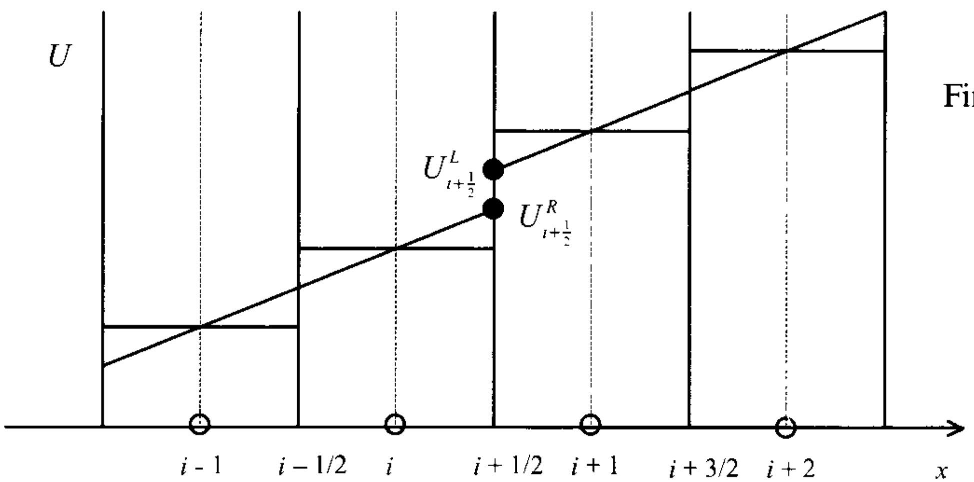
where the superscripts L and R refer to the left and right sides at the considered boundary and κ denotes a weight ($\kappa = -1, 0, 1$) leading to various extrapolation schemes Figure 6.2.5a,b).

The final solution is obtained as

$$\mathbf{U}^{n+1}_i = \mathbf{U}^n_i - \tau\left(\mathbf{f}^{**}_{i+\frac{1}{2}} - \mathbf{f}^{**}_{i-\frac{1}{2}}\right) \tag{6.2.72}$$



(a)



(b)

Figure 6.2.5 Variable extrapolation. **(a)** Piecewise linear representation within cells. **(b)** Linear one-sided extrapolation of interface values for $\kappa = -1$.

Second order upwind schemes in space and time

$$\mathbf{U}^{L*}_{i+\frac{1}{2}} = \overline{\mathbf{U}}_i + \frac{1}{4}[(1-\kappa)(\mathbf{U}_i - \mathbf{U}_{i-1}) + (1+\kappa)(\mathbf{U}_{i+1} - \mathbf{U}_i)]$$

$$\overline{\mathbf{U}}_i = \mathbf{U}^n_i - \frac{\Delta t}{2\Delta x}\left(\mathbf{f}^*_{i+\frac{1}{2}} - \mathbf{f}^*_{i-\frac{1}{2}}\right)$$

$$\mathbf{U}^{R*}_{i+\frac{1}{2}} = \overline{\mathbf{U}}_{i+1} - \frac{1}{4}[(1+\kappa)(\mathbf{U}_{i+1} - \mathbf{U}_i) + (1-\kappa)(\mathbf{U}_{i+2} - \mathbf{U}_{i+1})]$$

$$\overline{\mathbf{f}^{**}_{i+\frac{1}{2}}} = \frac{1}{2}\left[\mathbf{f}^*\left(\mathbf{U}^{L*}_{i+\frac{1}{2}}\right) + \mathbf{f}^*\left(\mathbf{U}^{R*}_{i+\frac{1}{2}}\right) - |\mathbf{a}|_{i+\frac{1}{2}}\left(\mathbf{U}^{R*}_{i+\frac{1}{2}} - \mathbf{U}^{L*}_{i+\frac{1}{2}}\right)\right]$$

Finally, we obtain

$$\mathbf{U}^{n+1}_i = \mathbf{U}^n_i - \tau\left(\overline{\mathbf{f}^{**}_{i+\frac{1}{2}}} - \overline{\mathbf{f}^{**}_{i-\frac{1}{2}}}\right)$$

SECOND ORDER UPWIND SCHEMES WITH HIGH RESOLUTION (TVD SCHEMES)

- *Entropy condition* – A decrease of entropy associated with expansion shocks must not be admitted.
- *Monotonicity condition* – This condition must be enforced to prevent oscillatory behavior in the numerical scheme.
- *Total Variation Diminishing (TVD)* – The total variation of any physically admissible solution must not be allowed to increase in time.

(b) Monotonicity Condition. A monotonicity condition refers to the nonoscillatory behavior of the numerical solution. Consider the solution of Euler equation to be in the form

$$u_i^{n+1} = H\left(u_{i-k}^n, u_{i-k+1}^n, \ldots, u_{i+k}^n\right) \tag{6.2.86}$$

This scheme is monotone if H is a monotonically increasing function such that

$$\frac{\partial H}{\partial u_j}(u_{i-k}, u_{i-k+1}, \ldots, u_{i+k}) \geq 0 \tag{6.2.87}$$

for all $i - k \leq j \leq i + k$, with

$$u_i^{n+1} = u_i^n - \tau \left(f_{i+\frac{1}{2}}^* - f_{i-\frac{1}{2}}^* \right) \tag{6.2.88}$$

$$f_{i+\frac{1}{2}}^* = f^* \left(u_{i-k+1}^n, u_{i-k+2}^n, \ldots, u_{i+k}^n \right) \tag{6.2.89}$$

The condition for monotonicity is given by

$$\frac{\partial f^*_{i+\frac{1}{2}}}{\partial u_{i-k+1}} \geq 0, \quad \frac{\partial f^*_{i+\frac{1}{2}}}{\partial u_{i+k}} \leq 0 \tag{6.2.90}$$

This represents a severe limitation, resulting in a scheme that is too diffusive. A compromise is the total variation diminishing concept, described next.

A numerical scheme is said to be total variation diminishing (TVD) if

$$TV(u^{n+1}) \leq TV(u^n) \qquad TV(U) = \sum_{j=-\infty}^{\infty} |U_{j+1} - U_j|$$

Let us consider the semi-discretized system

$$\frac{du_i}{dt} = -\frac{1}{\Delta x}\left(f^*_{i+\frac{1}{2}} - f^*_{i-\frac{1}{2}}\right)$$

TVD Schemes with Limiters

$$\frac{du_i}{dt} = -\frac{a^+}{\Delta x}\left[(u_i - u_{i-1}) + \frac{1}{2}(u_i - u_{i-1}) - \frac{1}{2}(u_{i-1} - u_{i-2})\right]$$
$$- \frac{a^-}{\Delta x}\left[(u_{i+1} - u_i) + \frac{1}{2}(u_{i+1} - u_i) - \frac{1}{2}(u_{i+2} - u_{i+1})\right]$$

Here, the variations in the second and third terms within the square brackets will be limited as follows:

$$\frac{du_i}{dt} = -\frac{a^+}{\Delta x}\left[(u_i - u_{i-1}) + \frac{1}{2}\Psi^+_{i-\frac{1}{2}}(u_i - u_{i-1}) - \frac{1}{2}\Psi^+_{i-\frac{3}{2}}(u_{i-1} - u_{i-2})\right]$$
$$- \frac{a^-}{\Delta x}\left[(u_{i+1} - u_i) + \frac{1}{2}\Psi^-_{i+\frac{1}{2}}(u_{i+1} - u_i) - \frac{1}{2}\Psi^-_{i+\frac{3}{2}}(u_{i+2} - u_{i+1})\right]$$

Now the TVD conditions are obtained by rewriting

$$\frac{du_i}{dt} = -\frac{a^+}{\Delta x}\left[1 + \frac{1}{2}\Psi^+_{i-\frac{1}{2}} - \frac{1}{2}\frac{\Psi_{i-\frac{3}{2}}}{r^+_{i-\frac{3}{2}}}\right](u_i - u_{i-1})$$

$$-\frac{a^-}{\Delta x}\left[1 + \frac{1}{2}\Psi^-_{i+\frac{1}{2}} - \frac{1}{2}\frac{\Psi^-_{i+\frac{3}{2}}}{r^-_{i+\frac{3}{2}}}\right](u_{i+1} - u_i)$$

with

$$r^+_{i+\frac{1}{2}} = \frac{u_{i+2} - u_{i+1}}{u_{i+1} - u_i}, \quad r^-_{i+\frac{1}{2}} = \frac{u_i - u_{i-1}}{u_{i+1} - u_i}$$

$$r^+_{i-\frac{1}{2}} = \frac{u_{i+1} - u_i}{u_i - u_{i-1}}, \quad r^-_{i-\frac{1}{2}} = \frac{u_{i-1} - u_{i-2}}{u_i - u_{i-1}}$$

$$r^+_{i-\frac{3}{2}} = \frac{u_i - u_{i-1}}{u_{i-1} - u_{i-2}}, \quad r^-_{i-\frac{3}{2}} = \frac{u_{i-2} - u_{i-3}}{u_{i-1} - u_{i-2}}$$

$$r^-_{i+\frac{3}{2}} = \frac{u_{i+3} - u_{i+2}}{u_{i+2} - u_{i+1}}, \quad r^-_{i+\frac{3}{2}} = \frac{u_{i+1} - u_i}{u_{i+2} - u_{i+1}}$$

$$\Psi^+_{i-\frac{1}{2}} = \Psi\left(r^+_{i-\frac{1}{2}}, r^+_{i+\frac{1}{2}}\right), \qquad \Psi^-_{i+\frac{1}{2}} = \Psi\left(r^-_{i-\frac{1}{2}}, r^-_{i+\frac{3}{2}}\right)$$

Thus, the TVD conditions are

$$\Psi^+ = 1 + \frac{1}{2}\Psi^+_{i-\frac{1}{2}} - \frac{1}{2}\frac{\Psi^+_{i-\frac{3}{2}}}{r^+_{i-\frac{3}{2}}} \geq 0$$

$$\Psi^- = 1 + \frac{1}{2}\Psi^-_{i+\frac{1}{2}} - \frac{1}{2}\frac{\Psi^-_{i+\frac{3}{2}}}{r^-_{i+\frac{3}{2}}} \geq 0$$

$$\frac{\Psi\left(r^+_{i-\frac{3}{2}}\right)}{r^+_{i-\frac{3}{2}}} - \Psi\left(r^+_{i-\frac{1}{2}}\right) \leq 2$$

$$\frac{\Psi\left(r^-_{i+\frac{3}{2}}\right)}{r^-_{i+\frac{3}{2}}} - \Psi\left(r^-_{i+\frac{1}{2}}\right) \leq 2$$

$$\Psi^+_{i-\frac{1}{2}} = \Psi\left(r^+_{i-\frac{1}{2}}\right), \qquad \Psi^+_{i-\frac{3}{2}} = \Psi\left(r^+_{i-\frac{3}{2}}\right)$$

$$\Psi^-_{i+\frac{1}{2}} = \Psi\left(r^-_{i+\frac{1}{2}}\right), \qquad \Psi^-_{i+\frac{3}{2}} = \Psi\left(r^-_{i+\frac{3}{2}}\right)$$

with the following constraints:

$$\Psi(r) \geq 0 \quad \text{for } r \geq 0$$

$$\Psi(r) = 0 \quad \text{for } r < 0$$

which may be generalized in the following form for all values of r and s:

$$\frac{\Psi(r)}{r} - \Psi(s) \leq 2$$

Thus, the sufficient condition becomes

$$0 \leq \Psi(r) \leq 2r$$

Various limiters for second order schemes are summarized below:

(a) TVD regions for $\Psi(r)$ in general

(b) Van Leer's limiter $\Psi = \dfrac{r + |r|}{1 + r}$

(c) Minimum modulus (minmod) $\Psi(r) = \begin{cases} \min(r, 1) & \text{if } r > 0 \\ 0 & \text{if } r \leq 0 \end{cases}$

$$\text{minmod}(x, y) = \begin{cases} x & \text{if } |x| < |y|, \quad xy > 0 \\ y & \text{if } |x| > |y|, \quad xy > 0 \\ 0 & \text{if } xy < 0 \end{cases}$$

(d) Roe's Superbee limiter $\Psi(r) = \max[0, \min(2r, 1), \min(r, 2)]$

(e) General β-limiters $\Psi = \max[0, \min(\beta r, 1), \min(r, \beta)], 1 \leq \beta \leq 2$

(f) Chakravarthy and Osher limiter $\Psi(r) = \max[0, \min(r, \beta)], 1 \leq \beta \leq 2$

In these limiters, we observe the following features:

(i) For $r < 1$ or

$$\frac{u_{i+1} - u_i}{\Delta x} < \frac{u_i - u_{i-1}}{\Delta x}$$

Then set $\Psi(r) = r$ and the contribution $u_i^n - u_{i-1}^n$ to u_i^{n+1} is replaced by the smaller quantity $(u_{i+1}^n - u_i^n)$.

(ii) If $r > 1$, the contribution $(u_i - u_{i-1})$ remains unchanged.

(iii) If the slopes of consecutive intervals change sign, then the updated point i receives no contribution from the upstream interval.

Explicit TVD Schemes of First Order Accuracy in Time.

$$\frac{du_i}{dt} = -\frac{1}{\Delta x}\left[1 + \frac{1}{2}\Psi\left(r^+_{i-\frac{1}{2}}\right) - \frac{1}{2}\frac{\Psi\left(r^+_{i-\frac{3}{2}}\right)}{r^+_{i-\frac{3}{2}}}\right]\left(f_i - f^*_{i-\frac{1}{2}}\right)^n$$
$$-\frac{1}{\Delta x}\left[1 + \frac{1}{2}\Psi\left(r^-_{i+\frac{1}{2}}\right) - \frac{1}{2}\frac{\Psi\left(r^-_{i+\frac{3}{2}}\right)}{r^-_{i+\frac{3}{2}}}\right]\left(f^*_{i+\frac{1}{2}} - f_i\right)^n$$

Define the local, positive, and negative CFL numbers,

$$\sigma^+_{i+\frac{1}{2}} = \tau\frac{f_{i+1} - f^*_{i+\frac{1}{2}}}{u_{i+1} - u_i} = \tau\frac{f^+_{i+1} - f^+_i}{u_{i+1} - u_i}$$

$$\sigma^-_{i+\frac{1}{2}} = \tau\frac{f^*_{i+\frac{1}{2}} - f_i}{u_{i+1} - u_i} = \tau\frac{f^-_{i+1} - f^-_i}{u_{i+1} - u_i}$$

with

$$\sigma_{i+\frac{1}{2}} = \sigma^+_{i+\frac{1}{2}} + \sigma^-_{i+\frac{1}{2}} = \tau\frac{f_{i+1} - f_i}{u_{i+1} - u_i} = \tau a_{i+\frac{1}{2}}$$

$$|\sigma|_{i+\frac{1}{2}} = \sigma^+_{i+\frac{1}{2}} - \sigma^-_{i+\frac{1}{2}} = \tau\left|\frac{f_{i+1} - f_i}{u_{i+1} - u_i}\right| = \tau|a|_{i+\frac{1}{2}}$$

$$\tau C^+_{i-\frac{1}{2}} = \sigma^+_{i-\frac{1}{2}}\left[1 + \frac{1}{2}\Psi\left(r^+_{i-\frac{1}{2}}\right) - \frac{1}{2}\frac{\Psi\left(r^+_{i-\frac{3}{2}}\right)}{r^+_{i-\frac{3}{2}}}\right]$$

$$\tau C^-_{i+\frac{1}{2}} = \sigma^-_{i+\frac{1}{2}}\left[1 + \frac{1}{2}\Psi\left(r^-_{i+\frac{1}{2}}\right) - \frac{1}{2}\frac{\Psi\left(r^-_{i+\frac{3}{2}}\right)}{r^-_{i+\frac{3}{2}}}\right]$$

Thus, the TVD condition

$$\tau\left(C^{+}_{i+\frac{1}{2}} - C^{-}_{i+\frac{1}{2}}\right) \le \tau|a|_{i+\frac{1}{2}}\left(\frac{1+\alpha}{2}\right) \le 1$$

where

$$\left|\Psi(s) - \frac{\Psi(r)}{r}\right| \le \alpha$$

with $0 < \alpha \le 2$. The CFL condition for this case is

$$|\sigma| \le \frac{2}{2+\alpha}$$

The stability conditions for various limiters are

minmod limiter: $|\sigma| < \frac{2}{3}$

superbee limiter: $|\sigma| < \frac{1}{2}$

and so on.

ESSENTIALLY NONOSCILLATORY SCHEME

In the ENO scheme, high-order accuracy is obtained, whenever the solution is smoothed by means of a piecewise polynomial reconstruction procedure, yielding high order pointwise information from the cell averages of the solution. When applied to piecewise smooth initial data, this reconstruction enables a flux computation which is of high order accuracy, whenever the function is smooth, and avoids nonconvergence.

The purpose of ENO is to achieve uniformly high order accuracy by avoiding the growth of spurious oscillations at shock discontinuities known as Gibb's phenomena. To this end, we employ piecewise polynomial reconstruction in the numerical solution based on an adaptive stencil. Such stencil is chosen according to the local smoothness of the flow variable.

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0$$

$$\overline{u}_i = \int_{x_i - 1/2}^{x_i + 1/2} u(\xi) d\xi \qquad \text{cell average } \overline{u}_i.$$

with $h_i = x_{i+1/2} - x_{i-1/2}$. Let us now reconstruct $u(x)$ from $\overline{u}_i$ by interpolating the primitive function $U(x)$,

$$U(x) = \int_{x_0}^{x} u(\xi) d\xi$$

Since we have

$$u(x) = \frac{d}{dx} U(x)$$

The point value of the primitive function at $x = x_{i+1/2}$ is given by

$$U_{i+1/2} = \sum_{i=i_0}^{i} \overline{u}_i h_i$$

reconstruction polynomial of the form

$$R(x, \overline{u}) = \frac{d}{dx} H_m(x, U)$$

where, for cell $x_{i-1/2}$ and $x_{i+1/2}$, $H_m(x, U)$ represents the mth degree polynomial that interpolates the values of $U_{i+1/2}$ at $m+1$ successive points $x_{j+1/2}(j_m \leq j \leq j_m + m)$ including $x_{i-1/2}$ and $x_{i+1/2}$. Thus, our objective is to choose a stencil with $H_m(x, U)$ being the smoothest. This can be extracted from a table of divided differences of $U(x)$