Lecture 4
Methods of solution of the finite difference equations. Parabolic equations-2

IMPLICIT SCHEMES

Laasonen Method

Contrary to the explicit schemes, the solution for *implicit schemes* involves the variables at more than one nodal point for the time step $(n+1)$. For example, we may write the difference equation for (4.2.1a) in the form

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)}{\Delta x^2}, \quad O(\Delta t, \Delta x^2) \tag{4.2.24}$$

This equation is written for all grid points at $n+1$ time step, leading to a tridiagonal form. The scheme given by (4.2.24) is known as the Laasonen method. This is unconditionally stable.

Crank-Nicolson Method

An alternative scheme of (4.2.24) is to replace the diffusion term by an average between n and $n+1$,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} + \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{\Delta x^2}\right], \quad O(\Delta t^2, \Delta x^2) \tag{4.2.25}$$

This may be rewritten as

$$A + B = C + D \tag{4.2.26}$$

where

$$A = \frac{u_i^{n+\frac{1}{2}} - u_i^n}{\Delta t/2}, \qquad B = \frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\Delta t/2}, \qquad C = \frac{\alpha\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right)}{(\Delta x)^2},$$

$$D = \frac{\alpha\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right)}{(\Delta x)^2}$$

Note that $A = C$ and $B = D$ represent explicit and implicit scheme, respectively. This scheme is known as the Crank-Nicolson method. It is seen that $A = C$ is solved explicitly for the time step $n + 1/2$ and the result is substituted into $B = D$. The scheme is unconditionally stable.

β-Method

A general form of the finite difference equation for (4.2.1) may be written as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\left[\frac{\beta\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)}{(\Delta x)^2} + \frac{(1-\beta)\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{(\Delta x)^2}\right] \tag{4.2.27}$$

This is known as the β-method. For $1/2 \leq \beta \leq 1$, the method is unconditionally stable. For $\beta = 1/2$, equation (4.2.27) reduces to the Crank-Nicolson scheme, whereas $\beta = 0$ leads to the FTCS method.

A numerical example for the solution of a typical parabolic equation characterized by Couette flow is presented in Section 4.7.2.

Let us now examine the solution of the two-dimensional diffusion equation,

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{4.2.28}$$

with the forward difference in time and the central difference in space (FTCS). We write an explicit scheme in the form

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha\left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right), \quad O(\Delta t, \Delta x^2, \Delta y^2) \tag{4.2.29}$$

It can be shown that the system is stable if

$$d_x + d_y \leq \frac{1}{2} \tag{4.2.30}$$

Here, diffusion numbers d_x and d_y are defined as

$$d_x = \frac{\alpha \Delta t}{\Delta x^2}, \qquad d_y = \frac{\alpha \Delta t}{\Delta y^2} \tag{4.2.31}$$

For simplicity, let $d_x = d_y = d$ for $\Delta x = \Delta y$. This will give $d \leq 1/4$ for stability, which is twice as restrictive. To avoid this restriction, consider an implicit scheme

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha\left(\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}\right) \tag{4.2.32}$$

or

$$d_x u_{i+1,j}^{n+1} + d_x u_{i-1,j}^{n+1} - (2d_x + 2d_y + 1)u_{i,j}^{n+1} + d_y u_{i,j+1}^{n+1} + d_y u_{i,j-1}^{n+1} = -u_{i,j}^{n} \tag{4.2.33}$$

This leads to a pentadiagonal system.

An alternative is to use the alternating direction implicit scheme, by splitting (4.2.25) into two equations:

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t/2} = \alpha\left(\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right) \tag{4.2.34a}$$

and

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\Delta t/2} = \alpha\left(\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2}\right) \tag{4.2.34b}$$

FRACTIONAL STEP METHODS

An approximation of multidimensional problems similar to ADI or approximate factorization schemes is also known as the method of fractional steps. This method splits the multidimensional equations into a series of one-dimensional equations and solves them sequentially. For example, consider a two-dimensional equation

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4.2.42}$$

The Crank-Nicolson scheme for (4.2.36) can be written in two steps:

$$\frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\frac{\Delta t}{2}} = \frac{\alpha}{2}\left[\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2} + \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2}\right] \tag{4.2.43a}$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = \frac{\alpha}{2}\left[\frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{\Delta y^2} + \frac{u_{i,j+1}^{n+\frac{1}{2}} - 2u_{i,j}^{n+\frac{1}{2}} + u_{i,j-1}^{n+\frac{1}{2}}}{\Delta y^2}\right] + O(\Delta t^2, \Delta x^2, \Delta y^2) \tag{4.2.43b}$$

This scheme is unconditionally stable.

The ADI method can be extended to three-space dimensions for the time intervals $n, n+1/3, n+2/3$, and $n+1$. Consider the unsteady diffusion problem,

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{4.2.44}$$

The three-step FDM equations are written as

$$\frac{u_{i,j,k}^{n+\frac{1}{3}} - u_{i,j,k}^{n}}{\Delta t/3} = \alpha\left(\frac{\delta_x^2 u_{i,j,k}^{n+\frac{1}{3}}}{\Delta x^2} + \frac{\delta_y^2 u_{i,j,k}^{n}}{\Delta y^2} + \frac{\delta_z^2 u_{i,j,k}^{n}}{\Delta z^2}\right) \tag{4.2.45a}$$

$$\frac{u_{i,j,k}^{n+\frac{2}{3}} - u_{i,j,k}^{n+\frac{1}{3}}}{\Delta t/3} = \alpha\left(\frac{\delta_x^2 u_{i,j,k}^{n+\frac{1}{3}}}{\Delta x^2} + \frac{\delta_y^2 u_{i,j,k}^{n+\frac{2}{3}}}{\Delta y^2} + \frac{\delta_z^2 u_{i,j,k}^{n+\frac{1}{3}}}{\Delta z^2}\right) \tag{4.2.45b}$$

$$\frac{u_{i,j,k}^{n+1} - u_{i,j,k}^{n+\frac{2}{3}}}{\Delta t/3} = \alpha\left(\frac{\delta_x^2 u_{i,j,k}^{n+\frac{2}{3}}}{\Delta x^2} + \frac{\delta_y^2 u_{i,j,k}^{n+\frac{2}{3}}}{\Delta y^2} + \frac{\delta_z^2 u_{i,j,k}^{n+1}}{\Delta z^2}\right), \quad O(\Delta t, \Delta x^2, \Delta y^2, \Delta z^2) \tag{4.2.45c}$$

This method is conditionally stable with $(d_x + d_y + d_z) \le 3/2$. A more efficient method may be derived using the Crank-Nicolson scheme.

$$\frac{u^*_{i,j,k} - u^n_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta^2_x u^*_{i,j,k} + \delta^2_x u^n_{i,j,k}}{\Delta x^2} + \frac{\delta^2_y u^n_{i,j,k}}{\Delta y^2} + \frac{\delta^2_z u^n_{i,j,k}}{\Delta z^2}\right]$$

$$\frac{u^{**}_{i,j,k} - u^n_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta^2_x u^*_{i,j,k} + \delta^2_x u^n_{i,j,k}}{\Delta x^2} + \frac{1}{2}\frac{\delta^2_y u^{**}_{i,j,k} + \delta^2_y u^n_{i,j,k}}{\Delta y^2} + \frac{\delta^2_z u^n_{i,j,k}}{\Delta z^2}\right]$$

$$\frac{u^{n+1}_{i,j,k} - u^n_{i,j,k}}{\Delta t} = \alpha\left[\frac{1}{2}\frac{\delta^2_x u^*_{i,j,k} + \delta^2_x u^n_{i,j,k}}{\Delta x^2} + \frac{1}{2}\frac{\delta^2_y u^{**}_{i,j,k} + \delta^2_y u^n_{i,j,k}}{\Delta y^2} + \frac{1}{2}\frac{\delta^2_z u^{n+1}_{i,j,k} + \delta^2_z u^n_{i,j,k}}{\Delta z^2}\right] \tag{4.2.46}$$

In this scheme, the final solution $u^{n+1}_{i,j,k}$ is obtained in terms of the intermediate steps $u^*_{i,j,k}$ and $u^{**}_{i,j,k}$.