Lecture 6. Methods of solution of the finite difference equations. Hyperbolic equations –nonlinear form

NONLINEAR PROBLEMS

A classical nonlinear first order hyperbolic equation is the Euler's equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} \tag{4.3.29}$$

which in conservation form may be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \tag{4.3.30a}$$

or

$$\frac{\partial u}{\partial t} = -\frac{\partial F}{\partial x} \quad \text{with } F = \left(\frac{u^2}{2}\right) \tag{4.3.30b}$$

Lax Method

In this method, the FTCS differencing scheme is used.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x}, \quad O(\Delta t, \Delta x^2) \tag{4.3.31}$$

To maintain stability, we replace u_i^n by its average,

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{2\Delta x}(F_{i+1}^n - F_{i-1}^n) \tag{4.3.32}$$

or

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\Delta t}{4\Delta x}\left[(u_{i+1}^n)^2 - (u_{i-1}^n)^2\right] \tag{4.3.33}$$

The solution will be stable if

$$\left|\frac{\Delta t}{\Delta x} u_{\max}\right| \le 1 \tag{4.3.34}$$

Beam-Warming Implicit Method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left[\left(\frac{\partial F}{\partial x}\right)_i^n + \left(\frac{\partial F}{\partial x}\right)_i^{n+1}\right] + O(\Delta t^2)$$

$$F^{n+1} = F^n + \frac{\partial F}{\partial u}\left(\frac{u^{n+1} - u^n}{\Delta t}\right)\Delta t + O(\Delta t^2)$$

with A being the Jacobian.

$$\left(\frac{\partial F}{\partial x}\right)^{n+1} = \left(\frac{\partial F}{\partial x}\right)^n + \frac{\partial}{\partial x}[A(u^{n+1} - u^n)]$$

$$A = \frac{\partial F}{\partial u} = \frac{\partial}{\partial u}\left(\frac{u^2}{2}\right) = u$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left\{\left(\frac{\partial F}{\partial x}\right)_i^n + \left(\frac{\partial F}{\partial x}\right)_i^n + \frac{\partial}{\partial x}[A(u_i^{n+1} - u_i^n)]\right\}$$

$$u_i^{n+1} = u_i^n - \frac{1}{2}\Delta t\left[\frac{2\left(F_{i+1}^n - F_{i-1}^n\right)}{2\Delta x} + \frac{A_{i+1}^n u_{i+1}^{n+1} - A_{i-1}^n u_{i-1}^{n+1}}{2\Delta x} - \frac{A_{i+1}^n u_{i+1}^n - A_{i-1}^n u_{i-1}^n}{2\Delta x}\right]$$

$$-\frac{\Delta t}{4\Delta x}A_{i-1}^n u_{i-1}^{n+1} + u_i^{n+1} + \frac{\Delta t}{4\Delta x}A_{i+1}^n u_{i+1}^{n+1}$$

$$= u_i^n - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right) + \frac{\Delta t}{4\Delta x}A_{i+1}^n u_{i+1}^n - \frac{\Delta t}{4\Delta x}A_{i-1}^n u_{i-1}^n + D$$

$$D = -\frac{\omega}{8}\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right), \qquad 0 < \omega < 1.$$

BURGERS' EQUATION

Consider the Burgers' equation written in various forms:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{4.4.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{4.4.2}$$

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{4.4.3}$$

FTCS Explicit Scheme

In this scheme (FTCS), approximations of forward differences in time and central differences in space are used:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \tag{4.4.4}$$

where the truncation error is $O(\Delta t, \Delta x^2)$. The central difference for the convective term tends to introduce significant damping.

DuFort-Frankel Explicit Scheme
In this scheme, we use second order central differences for all derivatives,

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + a\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu\frac{u_{i+1}^n - (u_i^{n-1} + u_i^{n+1}) + u_{i-1}^n}{\Delta x^2},$$

$$O\left(\Delta t^2, \Delta x^2, \left(\frac{\Delta t}{\Delta x}\right)^2\right) \qquad (4.4.7a)$$

or

$$u_i^{n+1} = \left(\frac{1-2d}{1+2d}\right)u_i^{n-1} + \left(\frac{C+2d}{1+2d}\right)u_{i-1}^n - \left(\frac{C-2d}{1+2d}\right)u_{i+1}^n \qquad (4.4.7b)$$

This is stable for $C \leq 1$.

MacCormack Implicit Scheme
One of the most frequently used implicit schemes is the MacCormack scheme.

Step 1

$$\left(1 + \lambda\frac{\Delta t}{\Delta x}\right)\delta u_i^* = \Delta u_i^n + \lambda\frac{\Delta t}{\Delta x}\delta u_{i+1}^*$$

$$u_i^* = u_i^n + \delta u_i^* \qquad (4.4.11a)$$

Step 2

$$\left(1 + \lambda\frac{\Delta t}{\Delta x}\right)\delta u_i^{n+1} = \Delta u_i^* + \lambda\frac{\Delta t}{\Delta x}\delta u_{i-1}^{n+1}$$

$$u_i^{n+1} = \frac{1}{2}(u_i^n + u_i^* + \delta u_i^{n+1}) \qquad (4.4.11b)$$

where

$$\lambda \geq \max\left[\frac{1}{2}\left(|a| + \frac{2\nu}{\Delta x} - \frac{\Delta x}{\Delta t}\right), 0\right] \qquad (4.4.12)$$