Lecture 1
Definition of the type of the partial differential equations and systems.

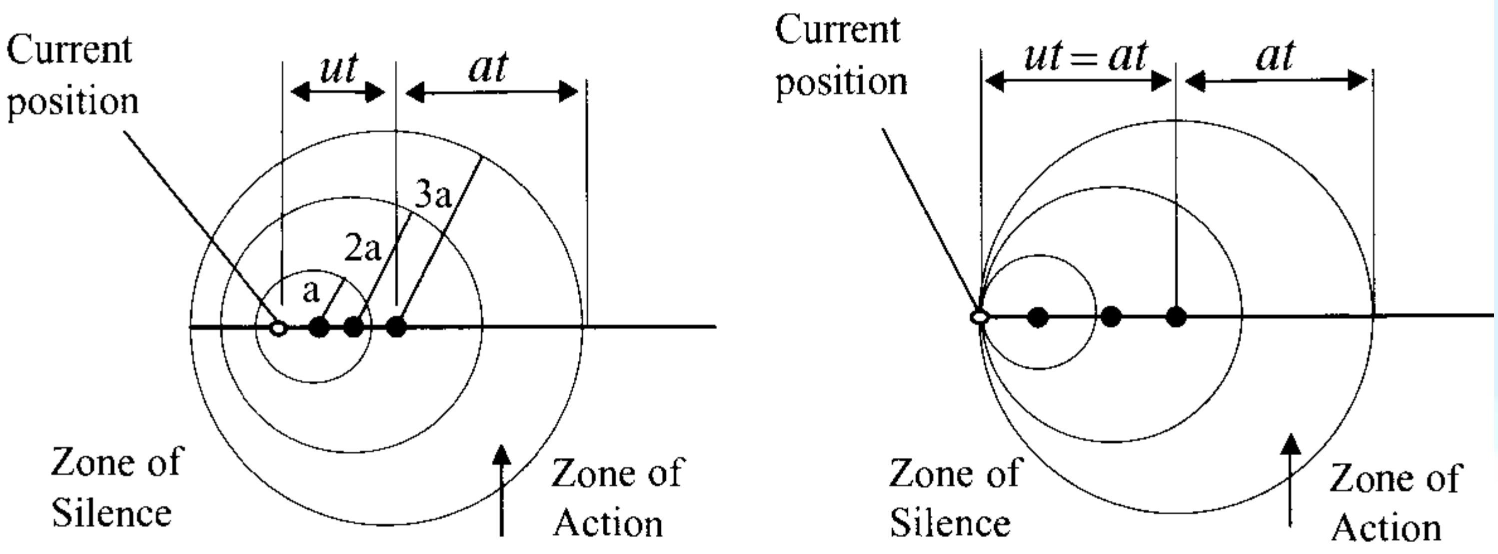
Partial differential equations (PDEs) in general, or the governing equations in fluid dynamics in particular, are classified into three categories: (1) elliptic, (2) parabolic, and (3) hyperbolic. The physical situations these types of equations represent can be illustrated by the flow velocity relative to the speed of sound as shown in Figure 1

Consider that the flow velocity u is the velocity of a body moving in the quiescent fluid. The movement of this body disturbs the fluid particles ahead of the body, setting off the propagation velocity equal to the speed of sound a. The ratio of these two competing speeds is defined as Mach number

$$M = \frac{u}{a}$$

For subsonic speed, $M < 1$, as time t increases, the body moves a distance, ut, which is always shorter than the distance at of the sound wave (The sound wave reaches the observer, prior to the arrival of the body, thus warning the observer that an object is approaching. The zones outside and inside of the circles are known as the zone of silence and zone of action, respectively.

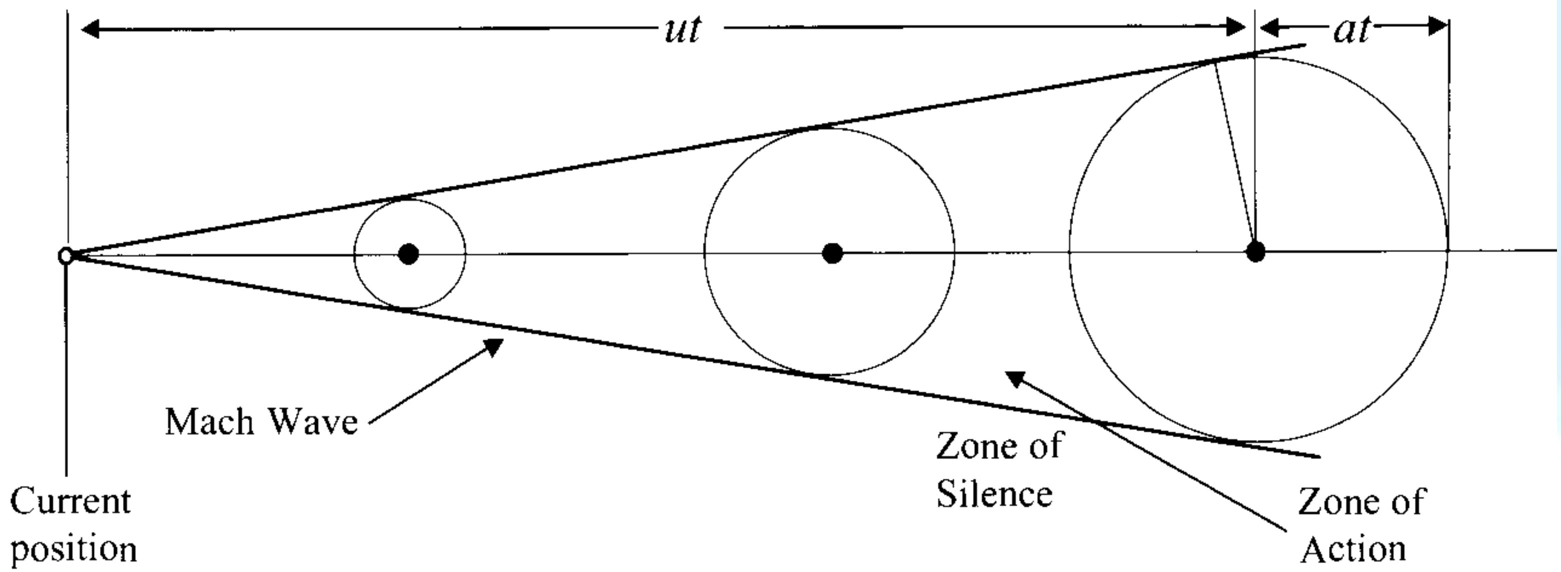
If, on the other hand, the body travels at the speed of sound, $M = 1$, then the observer does not hear the body approaching him prior to the arrival of the body, as these two actions are simultaneous All circles representing the distance traveled



(a) (b)

by the sound wave are tangent to the vertical line at the position of the observer. For supersonic speed, $M > 1$, the velocity of the body is faster than the speed of sound

The line tangent to the circles of the speed of sound, known as a Mach wave, forms the boundary between the zones of silence (outside) and action (inside). Only after the body has passed by does the observer become aware of it.



(c)

The governing equations for subsonic flow, transonic flow, and supersonic flow are classified as elliptic, parabolic, and hyperbolic, respectively. We shall elaborate on these equations below. Most of the governing equations in fluid dynamics are second order partial differential equations. For generality, let us consider the partial differential equation of the form [Sneddon, 1957] in a two-dimensional domain

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

where the coefficients A, B, C, D, E, and F are constants or may be functions of both independent and/or dependent variables. To assure the continuity of the first derivative of u, $u_x \equiv \partial u/\partial x$ and $u_y \equiv \partial u/\partial y$, we write

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy = \frac{\partial^2 u}{\partial x^2}dx + \frac{\partial^2 u}{\partial x \partial y}dy$$

$$du_y = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy = \frac{\partial^2 u}{\partial x \partial y}dx + \frac{\partial^2 u}{\partial y^2}dy$$

Here u forms a solution surface above or below the $x - y$ plane and the slope dy/dx representing the solution surface is defined as the characteristic curve.

Equations can be combined to form a matrix equation

$$\begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} H \\ du_x \\ du_y \end{bmatrix}$$

where

$$H = -\left(D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G\right)$$

Since it is possible to have discontinuities in the second order derivatives of the dependent variable along the characteristics, these derivatives are indeterminate. This happens when the determinant of the coefficient matrix in is equal to zero.

$$\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0$$

which yields

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

Solving this quadratic equation yields the equation of the characteristics in physical space,

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

The second order PDE is classified according to the sign of the expression $(B^2 - 4AC)$.

(a) Elliptic if $B^2 - 4AC < 0$
In this case, the characteristics do not exist.
(b) Parabolic if $B^2 - 4AC = 0$
In this case, one set of characteristics exists.
(c) Hyperbolic if $B^2 - 4AC > 0$
In this case, two sets of characteristics exist.

$$AX^2 + BXY + CY^2 + DX + EY + F = 0$$

in which one can identify the following geometrical properties:

$B^2 - 4AC < 0$ ellipse

$B^2 - 4AC = 0$ parabola

$B^2 - 4AC > 0$ hyperbola

Examples

(a) Elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1, \quad B = 0, \quad C = 1$$

$$B^2 - 4AC = -4 < 0$$

(b) Parabolic equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (\alpha > 0)$$

$$A = -\alpha, \quad B = 0, \quad C = 0$$

$$B^2 - 4AC = 0$$

(c) Hyperbolic equation

1-D First Order Wave Equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad (a > 0)$$

1-D Second Order Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$$

where

$$A = 1, \quad B = 0, \quad C = -a^2$$

$$B^2 - 4AC = 4a^2 > 0$$

2-D small disturbance potential equation

$$(1 - M^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$A = 1 - M^2, \quad B = 0, \quad C = 1$$

$$B^2 - 4AC = -4(1 - M^2)$$

elliptic $M < 1$

parabolic $M = 1$

hyperbolic $M > 1$

DIRICHLET BOUNDARY CONDITIONS

$$\frac{d^2u}{dx^2} - 2u = f(x) \quad 0 < x < 1$$

$$f(x) = 4x^2 - 2x - 4$$

subject to the Dirichlet boundary conditions:

$$u = 0 \qquad \text{at } x = 0$$
$$u = -1 \quad \text{at } x = 1$$

whose exact solution is given by $u = -2x^2 + x$.

NEUMANN BOUNDARY CONDITIONS

$$\frac{d^2u}{dx^2} - 2u = f(x) \quad 0 < x < 1$$

$$f(x) = 4x^2 - 2x - 4$$

subject to boundary conditions:

$$u = 0 \qquad \text{at } x = 0$$

$$\frac{du}{dx} = -3 \quad \text{at } x = 1$$

which has the exact solution:

$$u = -2x^2 + x$$

$$\alpha T + \beta \frac{\partial T}{\partial n} = \gamma$$

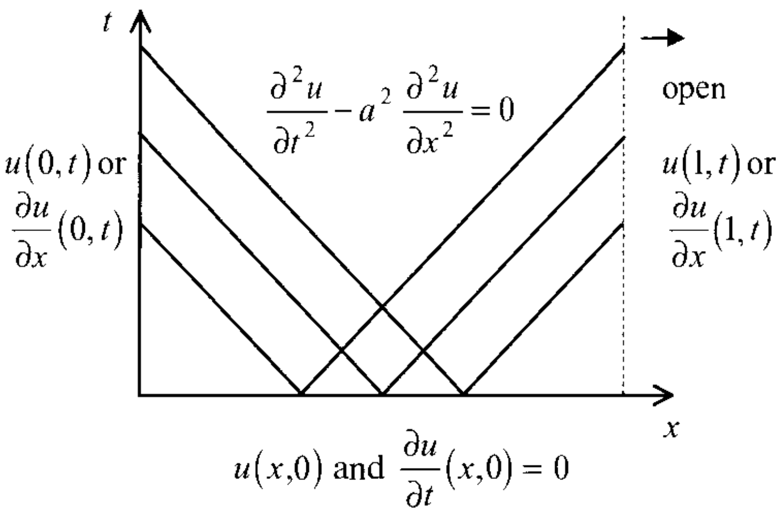
with

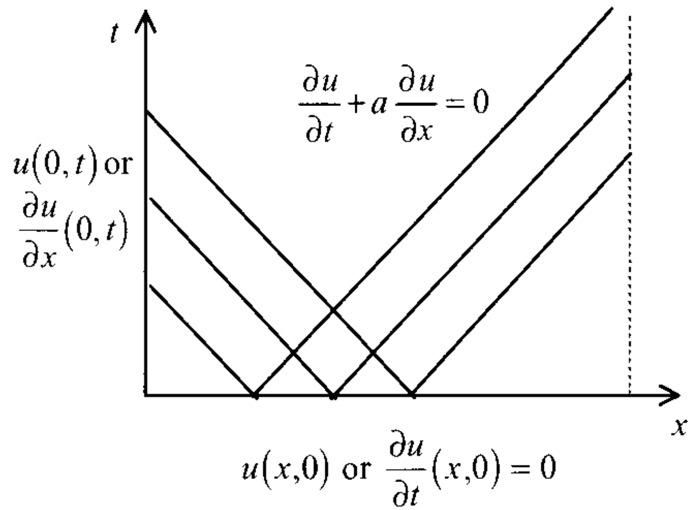
$$\frac{\partial T}{\partial n} = (\boldsymbol{n} \cdot \nabla) T = T_{,i} n_i = \frac{\partial T}{\partial x} n_1 + \frac{\partial T}{\partial y} n_2 + \frac{\partial T}{\partial z} n_3$$

$\beta = 0$	Dirichlet
$\alpha = 0$	Neumann
$\alpha \neq 0,\ \beta \neq 0$	Cauchy/Robin

For time dependent problems, we must provide initial conditions as well as boundary conditions. Let us consider the case of hyperbolic, parabolic, and elliptic equations as

(1) Hyperbolic equations





Second Order Equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad 0 < x < 1$$

Two initial conditions given $\left\{ u(x,0) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) \right.$

Two boundary conditions given $\left\{ u(0,t) \quad \text{or} \quad \frac{\partial u}{\partial x}(0,t) \right.$

$\left\{ u(1,t) \quad \text{or} \quad \frac{\partial u}{\partial x}(1,t) \right.$

First Order Equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad 0 < x < 1$$

One initial condition given $\left\{ u(x, 0) \quad \text{or} \quad \frac{\partial u}{\partial t}(x, 0) \right.$

One boundary condition given at $x = 0$ $\left\{ u(0, t) \quad \text{or} \quad \frac{\partial u}{\partial x}(0, t) \right.$

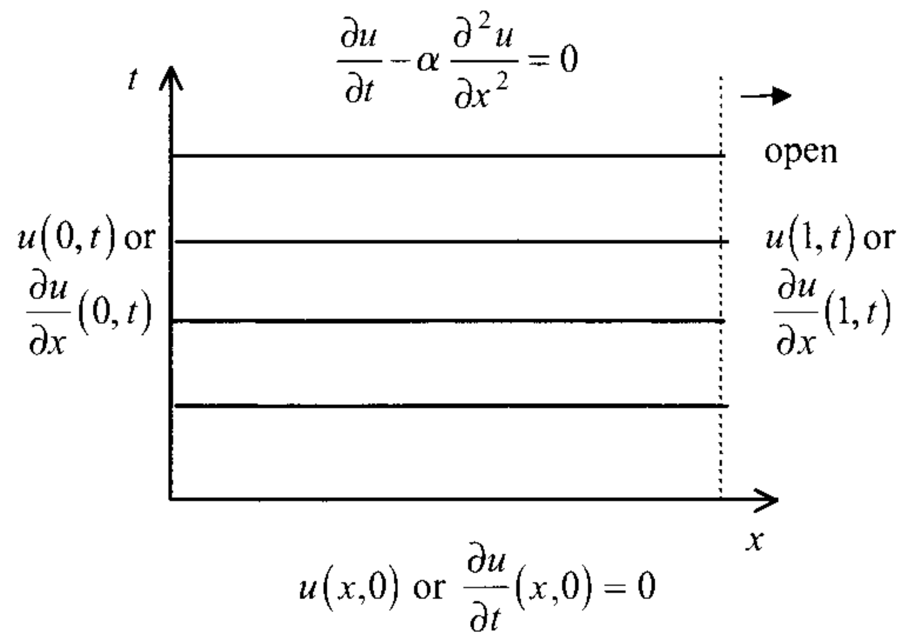
(2) Parabolic equations :

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \quad 0 < x < 1$$

One initial condition given $\left\{ u(x, 0) \quad \text{or} \quad \frac{\partial u}{\partial t}(x, 0) \right.$

Two boundary conditions given $\left\{ u(0, t) \quad \text{or} \quad \frac{\partial u}{\partial x}(0, t) \right.$

$\left\{ u(1, t) \quad \text{or} \quad \frac{\partial u}{\partial x}(1, t) \right.$



(b)

(3) Elliptic equations

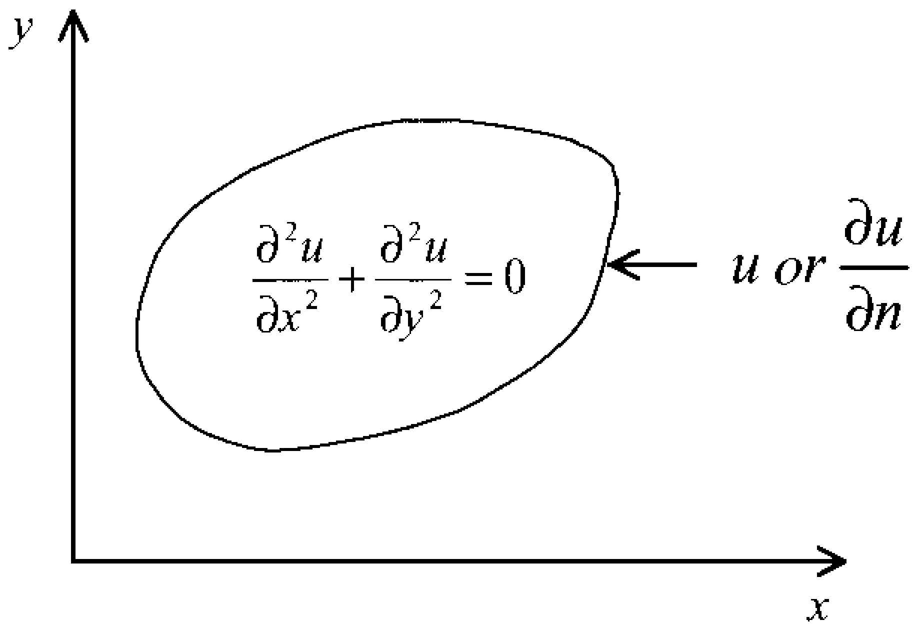
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad \Omega$$

Two boundary conditions given

$$u \quad \text{on} \quad \Gamma_D$$

$$\frac{\partial u}{\partial n} \quad \text{on} \quad \Gamma_N$$

where Γ_D and Γ_N denote the Dirichlet and Neumann boundaries,



The basic idea of finite difference methods is simple: derivatives in differential equations are written in terms of discrete quantities of dependent and independent variables, resulting in simultaneous algebraic equations with all unknowns prescribed at discrete mesh points for the entire domain.

In fluid dynamics applications, appropriate types of differencing schemes and suitable methods of solution are chosen, depending on the particular physics of the flows, which may include inviscid, viscous, incompressible, compressible, irrotational, rotational, laminar, turbulent, subsonic, transonic, supersonic, or hypersonic flows. Different forms of the finite difference equations are written to conform to these different physical phenomena encountered in fluid dynamics.

SIMPLE METHODS

Consider a function $u(x)$ and its derivative at point x,

$$\frac{\partial u(x)}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \tag{3.1.1}$$

If $u(x + \Delta x)$ is expanded in Taylor series about $u(x)$, we obtain

$$u(x + \Delta x) = u(x) + \Delta x \frac{\partial u(x)}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u(x)}{\partial x^2} + \frac{(\Delta x)^3}{3!} \frac{\partial^3 u(x)}{\partial x^3} + \cdots \tag{3.1.2}$$

Substituting (3.1.2) into (3.1.1) yields

$$\frac{\partial u(x)}{\partial x} = \lim_{\Delta x \to 0}\left(\frac{\partial u(x)}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u(x)}{\partial x^2} + \cdots\right) \tag{3.1.3}$$

Or it is seen from (3.1.2) that

$$\frac{u(x+\Delta x) - u(x)}{\Delta x} = \frac{\partial u(x)}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u(x)}{\partial x^2} + \cdots = \frac{\partial u(x)}{\partial x} + \mathrm{O}(\Delta x) \tag{3.1.4}$$

The derivative $\frac{\partial u(x)}{\partial x}$ in (3.1.4) is of first order in Δx, indicating that the truncation error $\mathrm{O}(\Delta x)$ goes to zero like the first power in Δx. The finite difference form given by (3.1.1), (3.1.3), and (3.1.4) is said to be of the first order accuracy.

we may write u in Taylor series at $i+1$ and $i-1$,

$$u_{i+1} = u_i + \Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{\Delta x^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \cdots \tag{3.1.5}$$

$$u_{i-1} = u_i - \Delta x\left(\frac{\partial u}{\partial x}\right)_i + \frac{\Delta x^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{\Delta x^3}{3!}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \frac{\Delta x^4}{4!}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \cdots \tag{3.1.6}$$

Rearranging (3.1.5), we arrive at the forward difference:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x) \tag{3.1.7}$$

Likewise, from (3.1.6), we have the backward difference:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x) \tag{3.1.8}$$

A central difference is obtained by subtracting (3.1.6) from (3.1.5):

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2) \tag{3.1.9}$$

It is seen that the truncation errors for the forward and backward differences are first order, whereas the central difference yields a second order truncation error.

Finally, by adding (3.1.5) and (3.1.6), we have

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = \left(\frac{\partial^2 u}{\partial x^2}\right)_i + \frac{(\Delta x)^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i + \cdots \tag{3.1.10}$$

This leads to the finite difference formula for the second derivative with second order accuracy,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2) \tag{3.1.11}$$