Lecture 10
Finite-difference methods of solutions of the incompressible flows-2.

VORTEX METHODS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),$$

$$\zeta = \mathrm{rot}\,\mathbf{q} \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \qquad u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\zeta,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial x\,\partial y}=$$

$$=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2}+\nu\frac{\partial}{\partial x}(\nabla^2 u),$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right)+\left(\frac{\partial v}{\partial y}\right)^2+v\frac{\partial^2 v}{\partial y^2}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x\,\partial y}=$$

$$=-\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2}+\nu\frac{\partial}{\partial y}(\nabla^2 v)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+$$

$$+u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}\right)+v\left(\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y^2}\right)=$$

$$=-\frac{1}{\rho}\nabla^2 p+\nu\left[\frac{\partial}{\partial x}(\nabla^2 u)+\frac{\partial}{\partial y}(\nabla^2 v)\right].$$

$$\nabla^2 p=2\rho\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right).$$

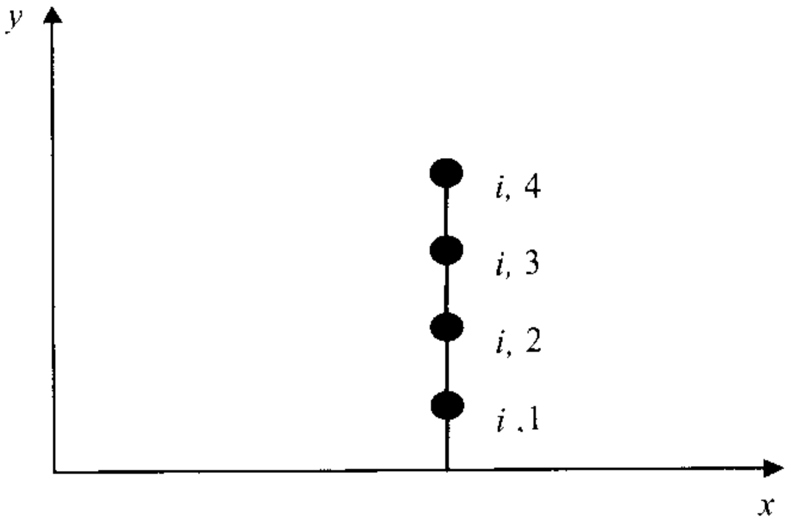
$$\nabla^2 p = S,$$

$$S = 2\rho\left[\left(\frac{\partial^2\psi}{\partial x^2}\right)\left(\frac{\partial^2\psi}{\partial y^2}\right) - \left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)^2\right].$$

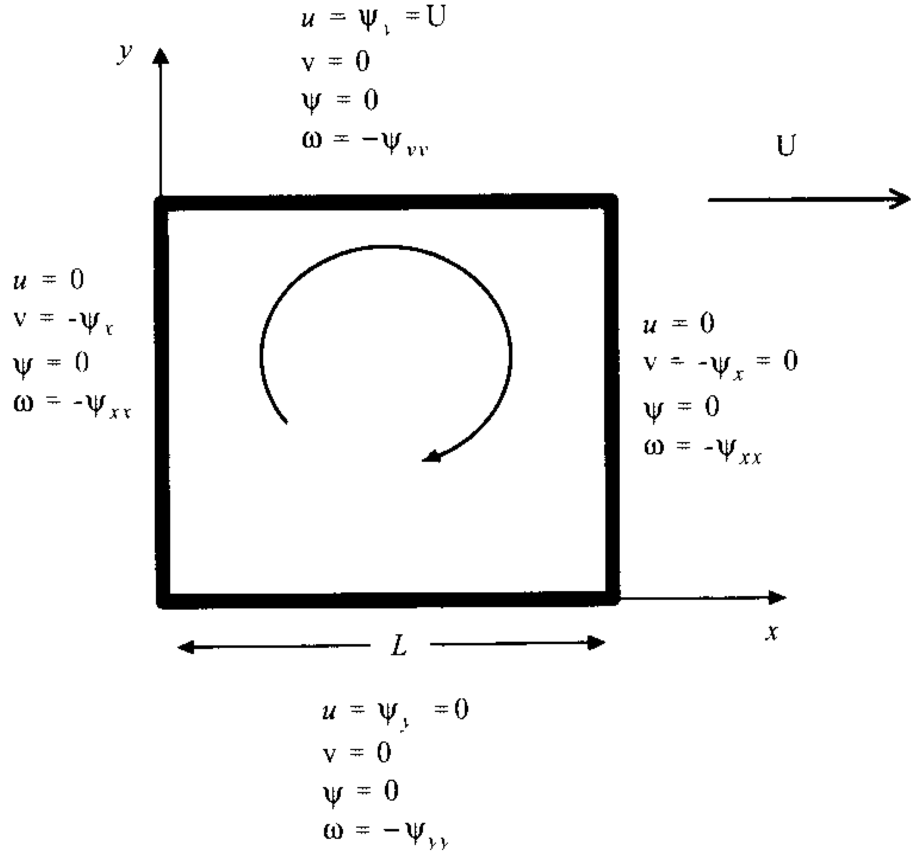
$$\frac{\partial p}{\partial x}\bigg|_w = \mu\,\frac{\partial^2 u}{\partial y^2}\bigg|_w, \qquad \frac{p_{i+1,1} - p_{i-1,1}}{2\Delta x} = -\mu\left(\frac{-3\zeta_{i,1} + 4\zeta_{i,2} - \zeta_{i,3}}{2\Delta y}\right).$$

$$\frac{\partial p}{\partial x}\bigg|_w = -\mu\,\frac{\partial \zeta}{\partial y}\bigg|_w,$$

$$S_{i,j} = 2\rho_{i,j}\left[\left(\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2}\right)\left(\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2}\right) - \left(\frac{\psi_{i+1,j+1} - \psi_{i+1,j-1} - \psi_{i-1,j+1} + \psi_{i-1,j-1}}{4\Delta x\,\Delta y}\right)^2\right]. \qquad (9.135)$$


y
i, 4
i, 3
i, 2
i, 1
x

(a)


$u = \psi_y = U$
$v = 0$
$\psi = 0$
$\omega = -\psi_{yy}$
y
U
$u = 0$
$v = -\psi_x$
$\psi = 0$
$\omega = -\psi_{xx}$
$u = 0$
$v = -\psi_x = 0$
$\psi = 0$
$\omega = -\psi_{xx}$
x
L
$u = \psi_y = 0$
$v = 0$
$\psi = 0$
$\omega = -\psi_{yy}$

(b)

Three dimensional vorticity transport equations

$$\boldsymbol{\psi} = \psi_x \mathbf{i} + \psi_y \mathbf{j} + \psi_z \mathbf{k},$$

$$u = \frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z},$$

$$v = -\frac{\partial \psi_z}{\partial x} + \frac{\partial \psi_x}{\partial z},$$

$$w = \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}.$$

$$\nabla \times (\nabla \times \boldsymbol{\psi}) = \boldsymbol{\zeta}.$$

$$\frac{\partial \zeta_x}{\partial t} + u\frac{\partial \zeta_x}{\partial x} + v\frac{\partial \zeta_x}{\partial y} + w\frac{\partial \zeta_x}{\partial z} - \zeta_x\frac{\partial u}{\partial x} - \zeta_y\frac{\partial u}{\partial y} - \zeta_z\frac{\partial u}{\partial z} = \nu\nabla^2\zeta_x,$$

$$\frac{\partial \zeta_y}{\partial t} + u\frac{\partial \zeta_y}{\partial x} + v\frac{\partial \zeta_y}{\partial y} + w\frac{\partial \zeta_y}{\partial z} - \zeta_x\frac{\partial v}{\partial x} - \zeta_y\frac{\partial v}{\partial y} - \zeta_z\frac{\partial v}{\partial z} = \nu\nabla^2\zeta_y,$$

$$\frac{\partial \zeta_z}{\partial t} + u\frac{\partial \zeta_z}{\partial x} + v\frac{\partial \zeta_z}{\partial y} + w\frac{\partial \zeta_z}{\partial z} - \zeta_x\frac{\partial w}{\partial x} - \zeta_y\frac{\partial w}{\partial y} - \zeta_z\frac{\partial w}{\partial z} = \nu\nabla^2\zeta_z.$$

Explicit methods

$$\zeta_i^{\overline{n+1}} = \zeta_i^n - \frac{C}{2}\left(\zeta_{i+1}^n - \zeta_{i-1}^n\right) + d\left(\zeta_{i+1}^n + \zeta_{i-1}^n - 2\zeta_i^{\overline{n+1}}\right),$$

$$\zeta_i^{n+1} = \zeta_i^n - \frac{C}{2}\left(\zeta_{i+1}^{\overline{n+1}} - \zeta_{i-1}^{\overline{n+1}}\right) + d\left(\zeta_{i+1}^{\overline{n+1}} + \zeta_{i-1}^{\overline{n+1}} - 2\zeta_i^{n+1}\right).$$

$C \leqslant 1$ и $d \leqslant 1/4$.

$$\frac{\zeta^* - \zeta^n}{\Delta t} = \alpha\frac{\delta^2}{\delta x^2}\left[\frac{1}{2}(\zeta^* + \zeta^n)\right] + \alpha\frac{\delta^2\zeta^n}{\delta y^2} + \alpha\frac{\delta^2\zeta^n}{\delta z^2}, \quad (3.312а)$$

$$\frac{\zeta^{**} - \zeta^n}{\Delta t} = \alpha\frac{\delta^2}{\delta x^2}\left[\frac{1}{2}(\zeta^* + \zeta^n)\right] + \alpha\frac{\delta^2}{\delta y^2}\left[\frac{1}{2}(\zeta^{**} + \zeta^n)\right] + \alpha\frac{\delta^2\zeta^n}{\delta z^2}, \quad (3.312б)$$

$O(\Delta t^2, \Delta x^2, \Delta y^2, \Delta z^2)$

$$\frac{\zeta^{n+1} - \zeta^n}{\Delta t} = \alpha\frac{\delta^2}{\delta x^2}\left[\frac{1}{2}(\zeta^* + \zeta^n)\right] + \alpha\frac{\delta^2}{\delta y^2}\left[\frac{1}{2}(\zeta^{**} + \zeta^n)\right] + \alpha\frac{\delta^2}{\delta z^2}\left[\frac{1}{2}(\zeta^{n+1} + \zeta^n)\right]. \quad (3.312в)$$

Implicit methods

$$\frac{\zeta^{n+1/2} - \zeta^n}{\Delta t/2} = -u\frac{\delta\zeta^{n+1/2}}{\delta x} - v\frac{\delta\zeta^n}{\delta y} + \alpha\frac{\delta^2\zeta^{n+1/2}}{\delta x^2} + \alpha\frac{\delta^2\zeta^n}{\delta y^2}, \quad (3.308а)$$

$$\frac{\zeta^{n+1} - \zeta^{n+1/2}}{\Delta t/2} = -u\frac{\delta\zeta^{n+1/2}}{\delta x} - v\frac{\delta\zeta^{n+1}}{\delta y} + \alpha\frac{\delta^2\zeta^{n+1/2}}{\delta x^2} + \alpha\frac{\delta^2\zeta^{n+1}}{\delta y^2} \quad (3.308б)$$

Alternating Directions Method - explicite

$$\zeta_i^{n+1} = \zeta_i^n + d\left(\zeta_{i+1}^n - \zeta_i^n - \zeta_i^{n+1} + \zeta_{i-1}^{n+1}\right) \quad \text{при } i\uparrow,$$

$$\zeta_i^{n+2} = \zeta_i^{n+1} + d\left(\zeta_{i+1}^{n+2} - \zeta_i^{n+2} - \zeta_i^{n+1} + \zeta_{i-1}^{n+1}\right) \quad \text{при } i\downarrow.$$

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} = -u\,\frac{\zeta_{i+1/2}^{n+1/2} - \zeta_{i-1/2}^{n+1/2}}{\Delta x}.$$

$$\zeta_{i+1/2}^{n+1/2} = \frac{1}{2}\left(\zeta_i^{n+1} + \zeta_{i+1}^n\right) \quad \text{при } i\uparrow,$$

$$\zeta_{i-1/2}^{n+1/2} = \frac{1}{2}\left(\zeta_{i-1}^{n+1} + \zeta_i^n\right) \quad \text{при } i\uparrow.$$

$$u\,\frac{\partial\zeta}{\partial x} = \alpha\,\frac{\partial^2\zeta}{\partial x^2}. \qquad u\,\frac{\zeta_{i+1} - \zeta_{i-1}}{2\,\Delta x} = \alpha\,\frac{\zeta_{i+1} - 2\zeta_i + \zeta_{i-1}}{\Delta x^2}.$$

$$\zeta_i^{k+1} = -\frac{u\,\Delta x}{4\alpha}\left(\zeta_{i+1}^k - \zeta_{i-1}^k\right) + \frac{1}{2}\left(\zeta_{i+1}^k + \zeta_{i-1}^k\right). \quad \mathrm{Re}_c \leqslant 2.$$

Methods for solutions the streamline function

$$\psi_{i,j}^{k+1} = \frac{1}{4}\left[\psi_{i+1,j}^k + \psi_{i-1,j}^k + \psi_{i,j+1}^k + \psi_{i,j-1}^k - \Delta^2\zeta_{i,j}\right].$$ *метод Ричардсона*

$$\psi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[\psi_{i+1,j}^k + \psi_{i-1,j}^k + \beta^2\psi_{i,j+1}^k + \beta^2\psi_{i,j-1}^k - \Delta x^2\zeta_{i,j}\right]. \quad \beta \doteq \Delta x/\Delta y;$$

$$\psi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[\psi_{i+1,j}^k + \psi_{i-1,j}^{k+1} + \beta^2\psi_{i,j+1}^k + \beta^2\psi_{i,j-1}^{k+1} - \Delta x^2\zeta_{i,j}\right],$$ *методом Либмана*

Метод релаксации невязки Саусвелла

$$r_{i,j} = \frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{\Delta x^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}}{\Delta y^2} - \zeta_{i,j}.$$

$$\psi_{i,j}^{k+1} = \frac{1}{2(1+\beta^2)}\left[\psi_{i+1,j}^k + \psi_{i-1,j}^k + \beta^2\psi_{i,j+1}^k + \beta^2\psi_{i,j-1}^k - \Delta x^2\zeta_{i,j}\right].$$

Alternating Directions Method - implicite

$$\psi^{k+1/2} = \psi^{k} + \frac{\alpha\,\Delta t}{2\,\Delta x^2}\left[\delta_x^2\psi^{k+1/2} + \beta^2\delta_y^2\psi^{k} + \Delta x^2\zeta\right],$$

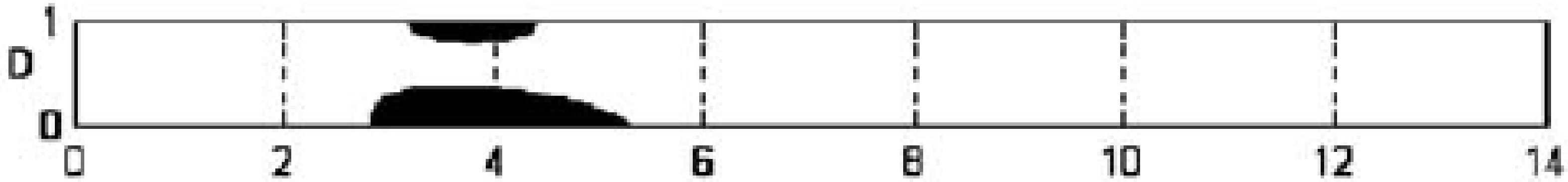
$$\psi^{k+1} = \psi^{k+1/2} + \frac{\alpha\,\Delta t}{2\,\Delta x^2}\left[\delta_x^2\psi^{k+1/2} + \beta^2\delta_y^2\psi^{k+1} + \Delta x^2\zeta\right],$$

$$\delta_x^2\psi = \psi_{i+1,\,j} - 2\psi_{i,\,j} + \psi_{i-1,\,j} \quad \text{и} \quad \delta_y^2\psi = \psi_{i,\,j+1} - 2\psi_{i,\,j} + \psi_{i,\,j-1}.$$

Pressure solution

$$P_{i,\,jc+1}^{k+1} = \frac{\omega}{2\,(1+\beta^2)}\left[P_{i+1,\,jc+1}^{k} + P_{i-1,\,jc+1}^{k+1} + \beta^2 P_{i,\,jc+2}^{k} + \beta^2 P_{i,\,jc}^{k} - \right.$$
$$\left. - \Delta x^2 S_{i,\,jc+1} - 2\,(1+\beta^2)\,P_{i,\,jc+1}^{k}\right] + P_{i,\,jc+1}^{k}$$

$$P_{i,\,jc+1}^{k+1} = \frac{\omega}{2\,(1+\beta^2)}\left[P_{i+1,\,jc+1}^{k} + P_{i-1,\,jc+1}^{k+1} + \beta^2 P_{i,\,jc+2}^{k} + \right.$$
$$+ \beta^2\left(P_{i,\,jc+1}^{k+1} - \left.\frac{\delta P}{\delta n}\right|_{i,\,jc}\Delta y\right) - \Delta x^2 S_{i,\,jc+1} -$$
$$\left. - 2\left(1+\beta^2\right)P_{i,\,jc+1}^{k}\right] + P_{i,\,jc+1}^{k},$$



$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial P}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right),$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{\partial P}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right),$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}=0.$$

$$\mathrm{Re}=\frac{U\cdot D}{\nu}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial P}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)-S\cdot(u-u_0).$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{\partial P}{\partial z}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-S\cdot(w-w_0),$$

$$S(x,z)=\begin{cases}0, & (x,z)\in D\\ \varepsilon^{-2}, & (x,z)\in D_0\end{cases}, \quad \varepsilon - \text{малый параметр},$$

$$\frac{\tilde{u}-u}{\tau}=-u\frac{\partial u}{\partial x}-w\frac{\partial u}{\partial z}+\frac{1}{R_e}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)-S\cdot(u-u_0),$$

$$\frac{\tilde{w}-w}{\tau}=-u\frac{\partial w}{\partial x}-w\frac{\partial w}{\partial z}+\frac{1}{R_e}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right)-S\cdot(w-w_0).$$

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial z^2}=\frac{1}{\tau}\left(\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{w}}{\partial z}\right).$$

$$\frac{u-\tilde{u}}{\tau}=\frac{\partial P}{\partial x},$$

$$\frac{w-\tilde{w}}{\tau}=\frac{\partial P}{\partial z}.$$

$$u(0,x,z)=u_H\cdot\left(\frac{z}{H}\right)^{1/2},\quad w(0,x,z)=0,\ 0\le x\le L,\ 0\le z\le H.$$

$x = 0,$

$$u(t,0,z)=u(0,0,z),\quad w(t,0,z)=0,\quad t>0,\ x=0,\ 0\le z\le H,$$

$$P(t,0,z)=P(0,0,z)=0,\quad t>0,\quad x=0,\ 0\le z\le H.$$

$x = L,$ $$\frac{\partial f}{\partial x} = 0, \quad f = (u, w, P), \quad x = L,$$

$$u(t,x,H) = u(0,0,H), \quad w(t,x,H) = 0, \quad t > 0, \quad z = H, \quad 0 \le x \le L,$$

$$P(t,x,H) = 0, \quad t > 0, \quad z = H, \quad 0 \le x \le L.$$

$z = 0$

$$u(t,x,0) = 0, \quad w(t,x,0) = 0, \quad t > 0, \quad z = 0, \quad 0 \le x \le L,$$

$$P(t,x,0) = 0, \quad t > 0, \quad z = 0, \quad 0 \le x \le L.$$

$$0{,}25(|u| + |w|)^2 \Delta t \mathrm{Re} \le 1 \text{ и } \Delta t / (\mathrm{Re}\, \Delta x^2) \le 0{,}25.$$