Lecture 8
Navier-Stokes system of equations.
Boundary layer.
Models of turbulence.

Boundary layer equations

$$u^* = \frac{u}{u_\infty}, \quad v^* = \frac{v}{u_\infty}, \quad x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad p^* = \frac{p}{\rho u_\infty^2},$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$

$$\mathrm{Ec} = 2(T_0 - T_\infty)/(T_w - T_\infty)$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0.$$

$$\mathrm{Re}_L = (\rho u_\infty L)/\mu \qquad \mathrm{Pr} = c_p \mu / k$$

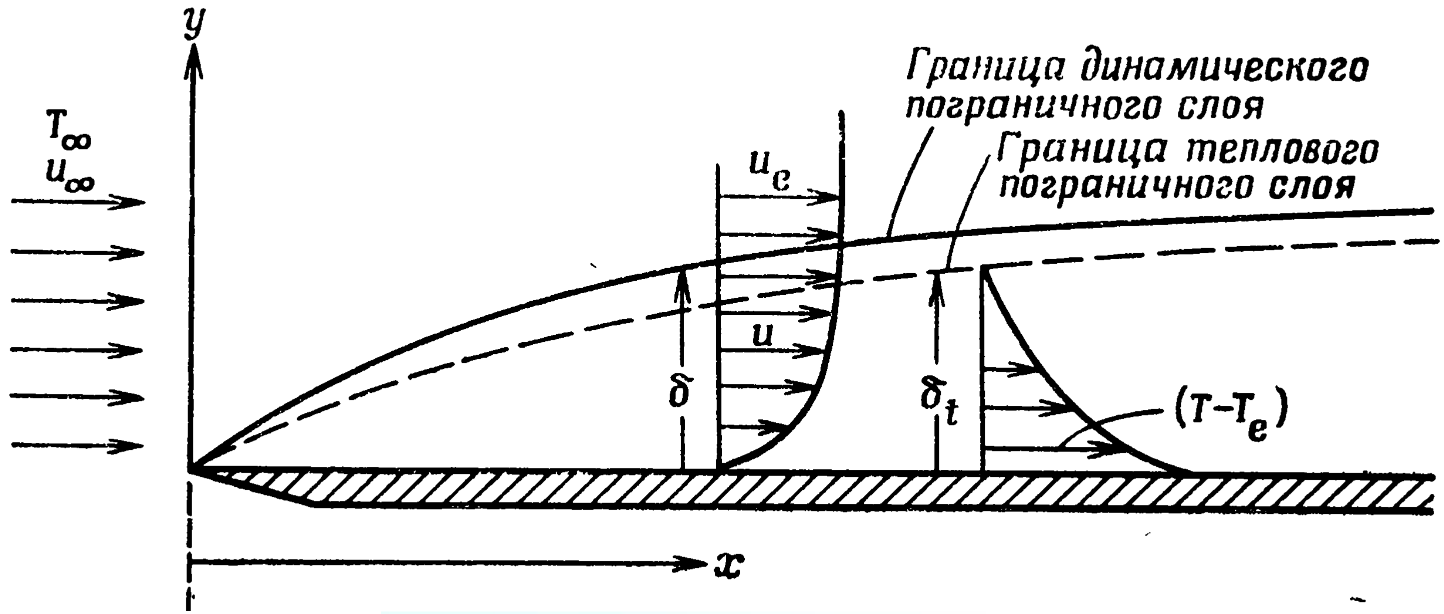
$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right).$$

$$u^* \frac{\partial v^*}{\partial x} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right).$$

$$u^* \frac{\partial \theta}{\partial x^*} + v^* \frac{\partial \theta}{\partial y^*} = \frac{1}{\mathrm{Re}_L \mathrm{Pr}}\left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}}\right) + \mathrm{Ec}\left(u^* \frac{\partial p^*}{\partial x^*} + v^* \frac{\partial p^*}{\partial y^*}\right) +$$

$$+ \frac{\mathrm{Ec}}{\mathrm{Re}_L}\left[2\left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\left(\frac{\partial v^*}{\partial y^*}\right)^2 + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*}\right)^2\right],$$

$$\delta/L \ll 1 \qquad \varepsilon = \delta/L$$



$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0.$$

1 1

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right).$$

1 1 ε 1/ε 1 ε^2 1 $1/\varepsilon^2$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right).$$

1 ε ε 1 ε ε^2 ε 1/ε

$$u^*\frac{\partial\theta}{\partial x^*}+v^*\frac{\partial\theta}{\partial y^*}=\frac{1}{\mathrm{Re}_L\,\mathrm{Pr}}\left(\frac{\partial^2\theta}{\partial x^{*2}}+\frac{\partial^2\theta}{\partial y^{*2}}\right)+\mathrm{Ec}\left(u^*\frac{\partial p^*}{\partial x^*}+v^*\frac{\partial p^*}{\partial y^*}\right)+$$

1 1 ε 1/ε ε² 1 1/ε² 1 1 1 ε ε

$$+\frac{\mathrm{Ec}}{\mathrm{Re}_L}\left[2\left(\frac{\partial u^*}{\partial x^*}\right)^2+2\left(\frac{\partial v^*}{\partial y^*}\right)^2+\left(\frac{\partial v^*}{\partial x^*}+\frac{\partial u^*}{\partial y^*}\right)^2\right].$$

ε² 1 1 ε² 1 1/ε²

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

$$\frac{dp}{dx}=-\rho u_e\frac{du_e(x)}{dx}.$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}.$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{\beta T u}{\rho c_p}\frac{dp}{dx}+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2,$$

$\nu=\mu/\rho$

$$u(x,0)=v(x,0)=0,$$

$$\beta=-\frac{1}{\rho}\frac{\partial\rho}{\partial T}\Big|_p.$$

$$T(x,0)=T_w(x)\quad\text{или}\quad\frac{\partial T}{\partial y}\Big|_{y=0}=\frac{q(x)}{k},$$

$$\lim_{y\to\infty}u(x,y)=u_e(x),\qquad\lim_{y\to\infty}T(x,y)=T_e(x),$$

Models of turbulence

Hypothesis concerning turbulent viscosity are considering as

I type models named as “turbulent viscosity model”.

II type models wich do not use Bossinesq hypothesis are named as “Reynolds stress models”

III type models are known as LES (large eddy simulation) models

Reynolds stress models

$$\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_j}(u_j u_k) = -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\tau_{jk}.$$

$$\frac{\partial}{\partial t}(\rho\overline{u'_i u'_k}) + \frac{\partial}{\partial x_j}(\rho\overline{u_j}\ \overline{u'_i u'_k}) + \frac{\partial}{\partial x_j}(\rho\ \overline{u'_i u'_j u'_k}) = -\ \overline{u'_i\frac{\partial p'}{\partial x_k}} -$$

$$-\ \overline{u'_k\frac{\partial p'}{\partial x_i}} + \overline{u'_i\frac{\partial \tau'_{jk}}{\partial x_j}} + \overline{u'_k\frac{\partial \tau'_{ji}}{\partial x_j}} - \rho\overline{u'_j u'_k}\frac{\partial u_i}{\partial x_j} - \rho\overline{u'_j u'_i}\frac{\partial u_k}{\partial x_j}.$$

$$\frac{\partial}{\partial t}(\overline{u'_i u'_k}) + \frac{\partial}{\partial x_j}(\overline{u_j}\ \overline{u'_i u'_k}) = -\frac{\partial}{\partial x_j}(\overline{u'_i u'_j u'_k}) + \nu\frac{\partial^2}{\partial x_j^2}(\overline{u'_i u'_k}) +$$

$$+\frac{1}{\rho}\overline{p'(\frac{\partial u'_i}{\partial x_k} + \frac{\partial u'_k}{\partial x_i})} - \left[\delta_{jk}\frac{\partial}{\partial x_j}(\overline{u'_i p'}) + \delta_{ij}\frac{\partial}{\partial x_j}(\overline{u'_k p'})\right] -$$

$$-\ 2\nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_k}{\partial x_j}} - \overline{u'_j u'_k}\frac{\partial u_i}{\partial x_j} - \overline{u'_j u'_i}\frac{\partial u_k}{\partial x_j}$$

$$\frac{\partial}{\partial t}(\overline{u'_i u'_k}) + \overline{u_j}\frac{\partial}{\partial x_j}(\overline{u'_i u'_k}) = \frac{\partial}{\partial x_j}D_{ik} + R_{ik} + P_{ik} - \varepsilon_{ik},$$

Algebraic models of turbulence

They are commonly based on Bossinesq hypothesis.
The most successful of them is Prandtle hypothesis

$$\mu_T = \rho l^2 \left| \frac{\partial u}{\partial y} \right|, \qquad \mu_T = \rho l^2 \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right]^{1/2}.$$

l is a characteristic length in crosscut direction

That formula gives us the viscosity coefficient as a scalar, and is useful for flux near the wall. More general cases can be described by the model

$$l_i = \varkappa y \left(1 - e^{-y^+/A^+} \right) \qquad l_0 = C_1 \delta,$$

$l = min(l_0, l_i)$ where $y^+ = \frac{y(|\tau_w|/\rho_w)^{1/2}}{\nu_w}$. C1≈0.089 $A^+=26$, $\kappa=0.41$,

$$Re_\theta = \rho_e u_e \theta / \mu_e \qquad \theta = \int_0^\infty \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e} \right) dy.$$

$$u^+ = u / (|\tau_w| / \rho_w)^{1/2}.$$

The one of the useful hypotesis is Prandtl and Kolmogorov theory, namely , they assumed that

$$\mu_T = \rho v_T l$$

v_T is proportional to the square root of kinetic energy of the turbulence:

$$\bar{k} = {}^1/_2\overline{u'_i u'_i}. \qquad \mu_T = C_k \rho l (\bar{k})^{1/2}$$

$$\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} D_s + P - \varepsilon_s,$$

$$D_s = \nu \frac{\partial k}{\partial x_j} - \frac{1}{\rho}\delta_{jk}\left(\overline{u'_k p'}\right) - \overline{u'_j k'} = D_{kk}/2; \quad k' = u'_k u'_k/2;$$

$$P = -\overline{u'_j u'_k}\frac{\partial \overline{u_k}}{\partial x_j} = P_{kk}/2; \qquad \varepsilon_s = \nu \overline{\frac{\partial u'_k}{\partial x_j}\frac{\partial u'_k}{\partial x_j}}.$$

$$-\frac{1}{\rho}\delta_{jk}\left(\overline{u'_k p'}\right) - \overline{u'_j k'} = \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j},$$

$$\varepsilon_s = c_D k^{3/2}/L,$$

$$\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \nu_t\left(\frac{\partial \overline{u}_j}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_j}\right)\frac{\partial \overline{u}_j}{\partial x_i} - c_D \frac{k^{3/2}}{L}.$$

$$\nu_t = c_\mu f_\mu k^2/\varepsilon.$$

$$\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] + \tau_{ij}\frac{\partial \overline{u_i}}{\partial x_j} - \varepsilon,$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j} + c_{\varepsilon 1}\frac{\varepsilon}{k}\tau_{ij}\frac{\partial \overline{u_i}}{\partial x_j} - c_{\varepsilon 2}\frac{\varepsilon^2}{k},$$

$$\nu_t = c_\mu k^2/\varepsilon,\; c_\mu = 0.09, c_{\varepsilon 1} = 1.44, c_{\varepsilon 2} = 1.92, \sigma_k = 1, \sigma_\varepsilon = 1.3.$$

$$\nu_t = k/\omega.$$

$$\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = \tau_{ij}\frac{\partial \overline{u_i}}{\partial x_j} - \beta^* k\omega + \frac{\partial}{\partial x_j}\left[(\nu + \nu_t)\frac{\partial k}{\partial x_j}\right].$$

$$\frac{\partial \omega}{\partial t} + \overline{u}_j \frac{\partial \omega}{\partial x_j} = \alpha\frac{\omega}{k}\tau_{ij}\frac{\partial \overline{u_i}}{\partial x_j} - \beta\omega^2 + \frac{\partial}{\partial x_j}\left[(\nu + \sigma\nu_t)\frac{\partial \omega}{\partial x_j}\right].$$

$$\alpha = \frac{13}{25}, \beta = \beta_o f_\beta, \beta^* = \beta_o^* f_{\beta^*}, \sigma = \frac{1}{2}, \sigma^* = \frac{1}{2},$$

$$\beta_o = \frac{9}{125}, f_\beta = \frac{1+70\chi_\omega}{1+80\chi_\omega}, \chi_\omega = \left|\frac{\Omega_{ij}\Omega_{jk}S_{ki}}{(\beta_o^*\omega)^3}\right|,$$

$$\beta_o^* = \frac{9}{100}, f_{\beta^*} = \begin{cases} 1, & \chi_k \le 0 \\ \frac{1+680\chi_k^2}{1+400\chi_k^2}, & \chi_k > 0 \end{cases}, \chi_k \equiv \frac{1}{\omega^3}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}.$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u_i}}{\partial x_j} - \frac{\partial \overline{u_j}}{\partial x_i}\right),$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right).$$

$\varepsilon = \beta^* \omega k$ и $l = k^{1/2}/\omega.$

LES (large eddy simulation) models

$$u'_i = u_i - \bar{u}_i \quad \text{и} \quad \triangle = (\triangle x \triangle y \triangle z)^{1/3}.$$

$$\bar{u}_i(\vec{x}, t) = \int\int\int G(\vec{x} - \vec{\xi}; \triangle) u_i(\vec{\xi}, t) d^3\vec{\xi}.$$

$$G(\vec{x} - \vec{\xi}; \triangle) = \begin{cases} 1/\triangle^3, & |x_i - \xi_i| < \triangle x_i/2 \\ 0, & |x_i - \xi_i| > \triangle x_i/2 \end{cases}$$

$$G(\vec{x} - \vec{\xi}; \triangle) = \frac{1}{\triangle^3} \Pi_{i=1}^{3} \frac{\sin(x_i - \xi_i)/\triangle}{(x_i - \xi_i)/\triangle}.$$

$$G(\vec{x} - \vec{\xi}; \triangle) = \left(\frac{6}{\pi \triangle^2}\right)^{3/2} \exp\left(-6 \frac{|\vec{x}_i - \vec{\xi}_i|^2}{\triangle^2}\right).$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0,$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{u_i u_j}\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k}.$$

$$\overline{u_i u_j} = \bar{u}_i \bar{u}_j + L_{ij} + C_{ij} + R_{ij},$$

$$L_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{u}_i \bar{u}_j, \qquad C_{ij} = \overline{\bar{u}_i u'_j} + \overline{\bar{u}_j u'_i}, \qquad R_{ij} = \overline{u'_i u'_j}.$$

$$\overline{\bar{u}_i} \neq \bar{u}_i,$$

$$L_{ij} \approx \frac{\gamma_l}{2} \nabla^2 (\bar{u}_i \bar{u}_j), \qquad \gamma_l = \int\int\int |\vec{\xi}|^2 G(\vec{\xi}) d^3 \bar{\xi}.$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j} (\overline{u_i u_j}) = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} [\nu \frac{\partial \bar{u}_i}{\partial x_j} + \tau_{ij}],$$

$$\tau_{ij} = -(Q_{ij} \quad \tau_{ij} = 2\nu_t S_{ij}, \quad S_{ij} = \frac{1}{2}(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}), = R_{ij} + C_{ij}.$$

$$\tau_{ij} = 2\nu_t S_{ij}, \quad S_{ij} = \frac{1}{2}(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}), \quad \nu_t = (C_s \triangle)^2 \sqrt{S_{ij} S_{ij}}$$

$$\tau_{ij} - (\delta_{ij}/3)\tau_{kk} = -2C\triangle^2 \underline{|S| S_{ij}} + L^m_{ij} - (\delta_{ij}/3) L^m_{kk},$$

$$|S| = \sqrt{S_{ij} S_{ij}} \qquad L^m_{ij} = \overline{\bar{u}_i \bar{u}_j} - \bar{\bar{u}}_i \bar{\bar{u}}_j \qquad C = -\frac{1}{2} \frac{L_{ij} M_{ij}}{M_{ij} M_{ij}},$$

$$L_{ij} = L^m_{ij} - \frac{1}{3} \delta_{ij} L^m_{kk}, \quad M_{ij} = \triangle^2 (\alpha^2 |\bar{S}| \bar{S}_{ij} - \overline{|S| S_{ij}}) - \frac{1}{3} \delta_{ij} M_{kk},$$