Lecture 5
Methods of solution of the finite difference equations. Hyperbolic equations – linear form.

Euler's Forward Time and Forward Space (FTFS) Approximations

Consider the first order wave equation (Euler equation) of the form

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a > 0 \tag{4.3.1}$$

The Euler's forward time and forward space approximation of (4.3.1) is written in the FTFS scheme as

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_{i+1}^n - u_i^n}{\Delta x} \tag{4.3.2}$$

It follows from (4.2.15) and (4.3.2) that the amplification factor assumes the form

$$g = 1 - C(e^{I\phi} - 1) = 1 - C(\cos\phi - 1) - IC\sin\phi = 1 + 2C\sin^2\frac{\phi}{2} - IC\sin\phi \tag{4.3.3}$$

with C being the Courant number or CFL number [Courant, Friedrichs, and Lewy, 1967],

$$C = \frac{a\Delta t}{\Delta x}$$

and

$$|g|^2 = g\,g^* = \left(1 + 2C\sin^2\frac{\phi}{2}\right)^2 + C^2\sin^2\phi = 1 + 4C(1-C)\sin^2\frac{\phi}{2} \geq 1 \tag{4.3.4}$$

where g^* is the complex conjugate of g. Note that the criterion $|g| \leq 1$ for all values of

Euler's Forward Time and Central Space (FTCS) Approximations

In this method, Euler's forward time and central space approximation of (4.3.1) is used:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x}, \quad O(\Delta t, \Delta x) \tag{4.3.5}$$

The von Neumann analysis shows that this is also unconditionally unstable.

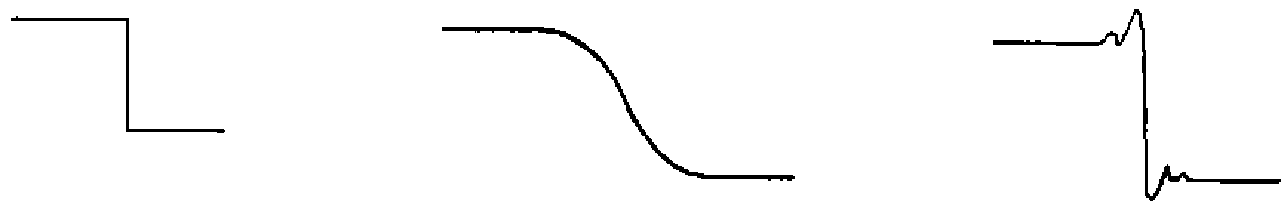
Euler's Forward Time and Backward Space (FTBS) Approximations – First Order Upwind Scheme

The Euler's forward time and backward space approximations (also known as up-wind method) is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\frac{u_i^n - u_{i-1}^n}{\Delta x}, \quad O(\Delta t, \Delta x) \tag{4.3.6}$$

The amplification factor takes the form

$$\begin{aligned} g &= 1 - C(1 - e^{-I\phi}) = 1 - C(1 - \cos\phi) - I\sin\phi \\ &= 1 - 2C\sin^2\frac{\phi}{2} - IC\sin\phi \end{aligned} \tag{4.3.7}$$



(1) Exact solution (2) Solution with dissipation error (3) Solution with dispersion error

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\,\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0, \quad c > 0.$$

$$u_t + cu_x = -\frac{\Delta t}{2} u_{tt} + \frac{c\Delta x}{2} u_{xx} - \frac{(\Delta t)^2}{6} u_{ttt} - c\frac{(\Delta x)^2}{6} u_{xxx} + \ldots.$$

$$u_{tt} + cu_{xt} = -\frac{\Delta t}{2} u_{ttt} + \frac{c\Delta x}{2} u_{xxt} - \frac{(\Delta t)^2}{6} u_{tttt} - \frac{c(\Delta x)^2}{6} u_{xxxt} + \ldots,$$

$$-cu_{tx} - c^2 u_{xx} = \frac{c\Delta t}{2} u_{ttx} - \frac{c^2\Delta x}{2} u_{xxx} + \frac{c(\Delta t)^2}{6} u_{tttx} + \\ + \frac{c^2(\Delta x)^2}{6} u_{xxxx} + \ldots.$$

$$u_{tt} = c^2 u_{xx} + \Delta t\left(\frac{-u_{ttt}}{2} + \frac{c}{2} u_{ttx} + O(\Delta t)\right) + \\ + \Delta x\left(\frac{c}{2} u_{xxt} - \frac{c^2}{2} u_{xxx} + O(\Delta x)\right).$$

$$u_{ttt} = -c^3 u_{xxx} + O(\Delta t,\ \Delta x),$$
$$u_{ttx} = c^2 u_{xxx} + O(\Delta t,\ \Delta x),$$
$$u_{xxt} = -cu_{xxx} + O(\Delta t,\ \Delta x).$$

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)\,u_{xx} - \frac{c(\Delta x)^2}{6}(2\nu^2 - 3\nu + 1)\,u_{xxx} + \\ + O((\Delta x)^3,\ (\Delta x)^2\Delta t,\ \Delta x(\Delta t)^2,\ (\Delta t)^3).$$

Lax Method

In this method, an average value of u_i^n in the Euler's FTCS is used:

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{C}{2}\left(u_{i+1}^n - u_{i-1}^n\right) \tag{4.3.13}$$

The von Neumann stability analysis shows that this scheme is stable for $C \le 1$.

$$u_t + cu_x = \frac{c\,\Delta x}{2}\left(\frac{1}{\nu} - \nu\right) u_{xx} + \frac{c\,(\Delta x)^2}{3}(1 - \nu^2)\, u_{xxx} + \ldots$$

Midpoint Leapfrog Method

Central differences for both time and spaces are used in this method:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -\frac{a\left(u_{i+1}^n - u_{i-1}^n\right)}{2\Delta x}, \quad O(\Delta t^2, \Delta x^2) \tag{4.3.14}$$

This scheme is stable for $C \le 1$. It has a second order accuracy, but requires two sets of initial values when the starter solution can provide only one set of initial data. This may lead to two independent solutions which are inaccurate.

$$u_t + cu_x = \frac{c\,(\Delta x)^2}{6}(\nu^2 - 1)\, u_{xxx} - \frac{c\,(\Delta x)^4}{120}(9\nu^4 - 10\nu^2 + 1)\, u_{xxxxx} + \ldots$$

Lax-Wendroff Method

In this method, we utilize the finite difference equation derived from Taylor series,

$$u(x, t + \Delta t) = u(x, t) + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + O(\Delta t^3) \tag{4.3.15a}$$

or

$$u_i^{n+1} = u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{1}{2!}\frac{\partial^2 u}{\partial t^2}\Delta t^2 + O(\Delta t^3) \tag{4.3.15b}$$

Differentiating (4.3.1) with respect to time yields

Substituting (4.3.1) and (4.3.16) into (4.3.15b) leads to

$$u_i^{n+1} = u_i^n + \Delta t\left(-a\frac{\partial u}{\partial x}\right) + \frac{\Delta t^2}{2}\left(a^2\frac{\partial^2 u}{\partial x^2}\right) \tag{4.3.17}$$

Using central differencing of the second order for the spatial derivative, we obtain

$$u_i^{n+1} = u_i^n - a\Delta t\left(\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right) + \frac{1}{2}(a\,\Delta t)^2\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right), \quad O(\Delta t^2, \Delta x^2) \tag{4.3.18}$$

This method is stable for $C \le 1$.

$$u_t + cu_x = -c\,\frac{(\Delta x)^2}{6}(1-\nu^2)\,u_{xxx} - \frac{c\,(\Delta x)^3}{8}\,\nu(1-\nu^2)\,u_{xxxx} + \ldots.$$

Lax-Wendroff Multistep Scheme

Step 1

$$u_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(u_{i+1}^n + u_i^n) - \frac{C}{2}(u_{i+1}^n - u_i^n), \quad O(\Delta t^2, \Delta x^2) \tag{4.3.25a}$$

Step 2

$$u_i^{n+1} = u_i^n - C\left(u_{i+\frac{1}{2}}^{n+\frac{1}{2}} - u_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right), \quad O(\Delta t^2, \Delta x^2) \tag{4.3.25b}$$

The stability condition is $C \le 1$. Note that substitution of (4.3.25a) into (4.3.25b) recovers the original Lax-Wendroff equation (4.3.18). The same result is obtained with (4.3.24a) and (4.3.24b).

MacCormack Multistep Scheme

Here we consider an intermediate step u_i^* which is related to $u_i^{n+\frac{1}{2}}$:

$$u_i^{n+\frac{1}{2}} = \frac{1}{2}(u_i^n + u_i^*) \tag{4.3.26}$$

Step 1

$$\frac{u_i^* - u_i^n}{\Delta t} = -a\frac{(u_{i+1}^n - u_i^n)}{\Delta x} \tag{4.3.27a}$$

Step 2

$$\frac{u_i^{n+1} - u_i^{n+\frac{1}{2}}}{\Delta t/2} = -a\frac{(u_i^* - u_{i-1}^*)}{\Delta x} \tag{4.3.27b}$$