Lecture 12
Finite difference methods for compressible flows. Navier-Stokes equations.

NAVIER-STOKES SYSTEM OF EQUATIONS

EXPLICIT SCHEMES

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial y} + \frac{\partial \mathbf{C}}{\partial z} = 0$$

with

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho u v - \tau_{xy} \\ \rho u w - \tau_{xz} \\ (\rho E + p)u - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} + q_x \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \rho v \\ \rho v u - \tau_{yx} \\ \rho v^2 + p - \tau_{yy} \\ \rho v w - \tau_{yz} \\ (\rho E + p)v - u\tau_{yx} - v\tau_{yy} - w\tau_{yz} + q_y \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \rho w \\ \rho w u - \tau_{zx} \\ \rho v w - \tau_{zy} \\ \rho w^2 + p - \tau_{zz} \\ (\rho E + p)w - u\tau_{zx} - v\tau_{zy} - w\tau_{zz} + q_z \end{bmatrix}$$

$$\tau_{xx} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right), \qquad \tau_{yy} = \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right),$$

$$\tau_{zz} = \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{yx}, \qquad \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \tau_{zx}, \qquad \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \tau_{zy}$$

$$q_x = -k\frac{\partial T}{\partial x}, \qquad q_y = -k\frac{\partial T}{\partial y}, \qquad q_z = -k\frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{U}}{J}\right) + \frac{\partial}{\partial \xi}\left[\frac{1}{J}(\xi_x\mathbf{A} + \xi_y\mathbf{B} + \xi_z\mathbf{C})\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{J}(\eta_x\mathbf{A} + \eta_y\mathbf{B} + \eta_z\mathbf{C})\right]$$
$$+ \frac{\partial}{\partial \zeta}\left[\frac{1}{J}(\zeta_x\mathbf{A} + \zeta_y\mathbf{B} + \zeta_z\mathbf{C})\right] = 0$$

$$J=[x_\xi(y_\eta z_\zeta - y_\zeta z_\eta) - x_\eta(y_\xi z_\zeta - y_\zeta z_\xi) - x_\zeta(y_\xi z_\eta - y_\eta z_\xi)]^{-1}$$

$$\xi_x = J(y_\eta z_\zeta - y_\zeta z_\eta), \qquad \xi_y = -J(x_\eta z_\zeta - x_\zeta z_\eta), \qquad \xi_z = J(x_\eta y_\zeta - x_\zeta y_\eta),$$

$$\eta_x = -J(y_\xi z_\zeta - y_\zeta z_\xi), \qquad \eta_y = J(x_\xi z_\zeta - x_\zeta z_\xi), \qquad \eta_z = -J(x_\xi y_\zeta - x_\zeta y_\xi),$$

$$\zeta_x = J(y_\xi z_\eta - y_\eta z_\xi), \qquad \zeta_y = -J(x_\xi z_\eta - x_\eta z_\xi), \qquad \zeta_z = J(x_\xi y_\eta - x_\eta y_\xi)$$

$$\tau_{xx} = \frac{2}{3}\mu[2(\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta) - (\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta) - (\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta)]$$

$$\tau_{yy} = \frac{2}{3}\mu[2(\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta) - (\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta) - (\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta)]$$

$$\tau_{zz} = \frac{2}{3}\mu[2(\xi_z w_\xi + \eta_z w_\eta + \zeta_z w_\zeta) - (\xi_x u_\xi + \eta_x u_\eta + \zeta_x u_\zeta) - (\xi_y v_\xi + \eta_y v_\eta + \zeta_y v_\zeta)]$$

$$\tau_{xy} = \mu(\xi_x u_\xi + \eta_y u_\eta + \zeta_y u_\zeta + \xi_x v_\xi + \eta_x v_\eta + \zeta_x v_\zeta)$$

$$\tau_{xz} = \mu(\xi_z u_\xi + \eta_z u_\eta + \zeta_z u_\zeta + \xi_x w_\xi + \eta_x w_\eta + \zeta_x w_\zeta)$$

$$\tau_{yz} = \mu(\xi_z v_\xi + \eta_y v_\eta + \zeta_y v_\zeta + \xi_y w_\xi + \eta_y w_\eta + \zeta_y w_\zeta)$$

$$q_x = -k(\xi_x T_\xi + \eta_x T_\eta + \zeta_x T_\zeta), \qquad q_x = -k(\xi_x T_\xi + \eta_x T_\eta + \zeta_x T_\zeta),$$

$$q_x = -k(\xi_x T_\xi + \eta_x T_\eta + \zeta_x T_\zeta)$$

Predictor

$$\overline{U_{i,j,k}^{n+1}} = \mathbf{U}_{i,j,k}^{n} - \frac{\Delta t}{\Delta x}\left(\mathbf{A}_{i+1,j,k}^{n} - \mathbf{A}_{i,j,k}^{n}\right) - \frac{\Delta t}{\Delta y}\left(\mathbf{B}_{i,j+1,k}^{n} - \mathbf{B}_{i,j,k}^{n}\right) - \frac{\Delta t}{\Delta z}\left(\mathbf{C}_{i,j,k+1}^{n} - \mathbf{C}_{i,j,k}^{n}\right) \tag{6.3.3}$$

Corrector

$$\mathbf{U}_{i,j,k}^{n+1} = \frac{1}{2}\Big[\mathbf{U}_{i,j,k}^{n} + \overline{U_{i,j,k}^{n+1}} - \frac{\Delta t}{\Delta x}\left(\overline{\mathbf{A}_{i+1,j,k}^{n+1}} - \overline{\mathbf{A}_{i,j,k}^{n+1}}\right) - \frac{\Delta t}{\Delta y}\left(\overline{\mathbf{B}_{i,j+1,k}^{n+1}} - \overline{\mathbf{B}_{i,j,k}^{n+1}}\right) - \frac{\Delta t}{\Delta z}\left(\overline{\mathbf{C}_{i,j,k+1}^{n+1}} - \overline{\mathbf{C}_{i,j,k}^{n+1}}\right)\Big] \tag{6.3.4}$$

$$\Delta t \le \frac{\sigma(\Delta t)_{\mathrm{CFL}}}{1 + 2/\mathrm{Re}_{\Delta}}$$

with $\sigma \cong 0.7 - 0.9$

$$(\Delta t)_{\mathrm{CFL}} \le \left[\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + \frac{|w|}{\Delta z} + a\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}\right]^{-1}$$

$$\mathrm{Re}_{\Delta} = \min(\mathrm{Re}_{\Delta x}, \mathrm{Re}_{\Delta y}, \mathrm{Re}_{\Delta z}) \ge 0$$

$$\mathrm{Re}_{\Delta x} = \frac{\rho|u|\Delta x}{\mu}, \qquad \mathrm{Re}_{\Delta y} = \frac{\rho|v|\Delta y}{\mu}, \qquad \mathrm{Re}_{\Delta z} = \frac{\rho|w|\Delta z}{\mu}$$

It is often necessary to add artificial viscosity using the fourth order derivatives of the form,

$$-\varepsilon(\Delta x_i \Delta x_j \Delta x_k \Delta x_m)\frac{\partial^4 \mathbf{U}}{\partial x_i \partial x_j \partial x_k \partial x_m}$$

where ε is an experimentally determined parameter.

IMPLICIT SCHEMES

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} = 0$$

$$\mathbf{a}_i = \frac{\partial \mathbf{F}_i}{\partial \mathbf{U}}, \qquad \mathbf{b}_i = \frac{\partial \mathbf{G}_i}{\partial \mathbf{U}}, \qquad \mathbf{c}_{ij} = \frac{\partial \mathbf{G}_i}{\partial \mathbf{U}_{,j}}$$

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho \mathrm{v} \\ \rho E \end{bmatrix} = \begin{bmatrix} \rho \\ \ell \\ m \\ e \end{bmatrix}$$

$$\mathbf{F}_1 = \begin{bmatrix} F_1^1 \\ F_1^2 \\ F_1^3 \\ F_1^4 \end{bmatrix} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho u \mathrm{v} \\ \rho E u + p u \end{bmatrix} = \begin{bmatrix} \ell \\ p + \ell^2/\rho \\ \ell m/\rho \\ (p+e)\dfrac{\ell}{\rho} \end{bmatrix}$$

$$\mathbf{F}_2 = \begin{bmatrix} F_2^1 \\ F_2^2 \\ F_2^3 \\ F_2^4 \end{bmatrix} = \begin{bmatrix} \rho \mathrm{v} \\ \rho \mathrm{v} u \\ p + \rho \mathrm{v}^2 \\ \rho E \mathrm{v} + p \mathrm{v} \end{bmatrix} = \begin{bmatrix} m \\ \ell m/\rho \\ p + m^2/\rho \\ (p+e)\dfrac{m}{\rho} \end{bmatrix}$$

where

$$p = (\gamma - 1)\rho\left(E - \frac{1}{2}\mathrm{v}_j \mathrm{v}_j\right) = (\gamma - 1)\rho\left[E - \frac{1}{2}(u^2 + \mathrm{v}^2)\right] = (\gamma - 1)\left[e - \frac{1}{2\rho}(\ell^2 + m^2)\right]$$

$$T = \frac{1}{c_v}\left(E - \frac{1}{2}\mathrm{v}_j \mathrm{v}_j\right) = \frac{1}{c_v}\left[E - \frac{1}{2}(u^2 + \mathrm{v}^2)\right] = \frac{1}{\rho c_v}\left[e - \frac{1}{2\rho}(\ell^2 + m^2)\right]$$

The convective Jacobian $\mathbf{a}_i$ can be evaluated as

$$\mathbf{a}_1 = \frac{\partial \mathbf{F}_1}{\partial \mathbf{U}} = \begin{bmatrix} \frac{\partial F_1^1}{\partial U_1} & \frac{\partial F_1^1}{\partial U_2} & \frac{\partial F_1^1}{\partial U_3} & \frac{\partial F_1^1}{\partial U_4} \\ \frac{\partial F_1^2}{\partial U_1} & \frac{\partial F_1^2}{\partial U_2} & \frac{\partial F_1^2}{\partial U_3} & \frac{\partial F_1^2}{\partial U_4} \\ \frac{\partial F_1^3}{\partial U_1} & \frac{\partial F_1^3}{\partial U_2} & \frac{\partial F_1^3}{\partial U_3} & \frac{\partial F_1^3}{\partial U_4} \\ \frac{\partial F_1^4}{\partial U_1} & \frac{\partial F_1^4}{\partial U_2} & \frac{\partial F_1^4}{\partial U_3} & \frac{\partial F_1^4}{\partial U_4} \end{bmatrix}$$

$$\mathbf{a}_1 = \frac{\partial \mathbf{F}_1}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma - 3}{2}u^2 + \frac{\gamma - 1}{2}\mathrm{v}^2 & (3-\gamma)u & -(\gamma - 1)\mathrm{v} & \gamma - 1 \\ -u\mathrm{v} & \mathrm{v} & u & 0 \\ -\frac{\gamma e u}{\rho} + (\gamma - 1)u(u^2 + \mathrm{v}^2) & \frac{\gamma e}{\rho} + \frac{1-\gamma}{2}(3u^2 + \mathrm{v}^2) & (1-\gamma)u\mathrm{v} & \gamma u \end{bmatrix}$$

$$\mathbf{a}_2 = \frac{\partial \mathbf{F}_2}{\partial \mathbf{U}}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -u\mathrm{v} & \mathrm{v} & u & 0 \\ \frac{\gamma-3}{2}\mathrm{v}^2 + \frac{\gamma-1}{2}u^2 & -(\gamma-1)u & (3-\gamma)\mathrm{v} & \gamma-1 \\ -\frac{\gamma e \mathrm{v}}{\rho} + (\gamma-1)\mathrm{v}(u^2+\mathrm{v}^2) & (1-\gamma)u\mathrm{v} & \frac{\gamma e}{\rho} + \frac{1-\gamma}{2}(3\mathrm{v}^2+u^2) & \gamma \mathrm{v} \end{bmatrix}$$

$$\mathbf{G}_1 = \begin{bmatrix} G_1^1 \\ G_1^2 \\ G_1^3 \\ G_1^4 \end{bmatrix} = -\begin{bmatrix} 0 \\ \tau_{11} \\ \tau_{12} \\ \tau_{11}u + \tau_{12}\mathrm{v} - q_1 \end{bmatrix} \qquad \mathbf{G}_2 = \begin{bmatrix} G_2^1 \\ G_2^2 \\ G_2^3 \\ G_2^4 \end{bmatrix} = -\begin{bmatrix} 0 \\ \tau_{21} \\ \tau_{22} \\ \tau_{21}u + \tau_{22}\mathrm{v} - q_2 \end{bmatrix}$$

$$\mathbf{b}^1 = \frac{\partial \mathbf{G}_1}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ b_{21}^1 & b_{22}^1 & b_{23}^1 & 0 \\ b_{31}^1 & b_{32}^1 & b_{33}^1 & 0 \\ b_{41}^1 & b_{42}^1 & b_{43}^1 & b_{44}^1 \end{bmatrix} \qquad \mathbf{b}_2 = \frac{\partial \mathbf{G}_2}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ b_{21}^2 & b_{22}^2 & b_{23}^2 & 0 \\ b_{31}^2 & b_{32}^2 & b_{33}^2 & 0 \\ b_{41}^2 & b_{42}^2 & b_{43}^2 & b_{44}^2 \end{bmatrix}$$

with $m_1 = \rho u, m_2 = \rho v$

$$b^1_{21} = -\frac{1}{\rho^2}\left(-\mu_R m_{1,1} - \lambda m_{2,2} + 2\mu_R \frac{m_1\rho_{,1}}{\rho} + 2\lambda \frac{m_2\rho_{,2}}{\rho}\right) \quad b^1_{22} = \frac{\mu_R}{\rho^2}\rho_{,1} \quad b^1_{23} = \frac{\lambda}{\rho^2}\rho_{,2}$$

$$b^1_{31} = -\frac{\mu}{\rho^2}\left(-m_{2,1} - m_{1,2} + 2\frac{m_1\rho_{,2}}{\rho} + 2\frac{m_2\rho_{,1}}{\rho}\right) \qquad b^1_{32} = \frac{\mu}{\rho^2}\rho_{,2} \quad b^1_{33} = \frac{\mu}{\rho^2}\rho_{,1}$$

$$b^1_{41} = ub^1_{21} + vb^1_{31} - \frac{1}{\rho^2}(m_1\tau_{11} + m_2\tau_{12}) + \frac{k}{\rho^2 c_v}[-(\rho E)_{,1} + 2um_{1,1}$$
$$+ 2vm_{2,1} + (2E - 3u^2 - 3v^2)\rho_{,1}]$$

$$b^1_{42} = -\frac{\tau_{11}}{\rho} + ub^1_{22} + vb^1_{32} + \frac{k}{\rho^2 c_v}[-m_{1,1} + 2u\rho_{,1}]$$

$$b^1_{43} = -\frac{\tau_{12}}{\rho} + ub^1_{23} + vb^1_{33} - \frac{k}{\rho^2 c_v}[-m_{2,1} + 2v\rho_{,1}] b^1_{44} = \frac{k}{\rho^2 c_v}\rho_{,1}$$

$$b^2_{21} = b^1_{31}, \quad b^2_{22} = b^1_{32}, \quad b^2_{23} = b^1_{33}, \quad b^2_{31} = \frac{1}{\rho^2}(\lambda m_{1,1} + \mu_R m_{2,2} + \mu_R u\rho_{,1} - \mu_R v\rho_{,2}),$$

$$b^2_{32} = \frac{\lambda}{\rho^2}\rho_{,1}, \quad b^2_{33} = \frac{\mu_R}{\rho^2}\rho_{,2}$$

$$b^2_{41} = ub^2_{21} + vb^2_{31} + \frac{1}{\rho^2}(m_1\tau_{12} + m_2\tau_{22}) - \frac{k}{\rho^2 c_v}[-(\rho E)_{,2} + 2um_{1,2}$$
$$+ 2vm_{2,2} + (2E - 3u^2 - 3v^2)\rho_{,2}],$$

$$b_{42}^2 = -\frac{\tau_{12}}{\rho} + ub_{22}^2 - \mathrm{v}b_{32}^2 + \frac{k}{\rho^2 c_v}[-m_{1,2} + 2u\rho_{,2}],$$

$$b_{43}^2 = -\frac{\tau_{22}}{\rho} + ub_{23}^2 + \mathrm{v}b_{33}^2 - \frac{k}{\rho^2 c_v}[-m_{2,2} + 2\mathrm{v}\rho_{,2}], \quad b_{44}^2 = \frac{k}{\rho^2 c_v}\rho_{,2}$$

The diffusion gradient Jacobians are evaluated as

$$\mathbf{c}_{11} = \frac{\partial \mathbf{G}_1}{\partial \mathbf{U}_{,1}} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_{21}^{11} & c_{21}^{11} & 0 & 0 \\ c_{31}^{11} & 0 & c_{33}^{11} & 0 \\ c_{41}^{11} & c_{42}^{11} & c_{43}^{11} & c_{44}^{11} \end{bmatrix} \qquad \mathbf{c}_{12} = \frac{\partial \mathbf{G}_1}{\partial \mathbf{U}_{,2}} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_{21}^{12} & 0 & c_{23}^{12} & 0 \\ c_{31}^{13} & c_{32}^{12} & 0 & 0 \\ c_{41}^{12} & c_{42}^{12} & c_{43}^{12} & 0 \end{bmatrix}$$

$$\mathbf{c}_{21} = \frac{\partial \mathbf{G}_2}{\partial \mathbf{U}_{,1}} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_{21}^{21} & 0 & c_{23}^{21} & 0 \\ c_{31}^{21} & c_{32}^{21} & 0 & 0 \\ c_{41}^{21} & c_{42}^{21} & c_{43}^{21} & 0 \end{bmatrix} \qquad \mathbf{c}_{22} = \frac{\partial \mathbf{G}_2}{\partial \mathbf{U}_{,2}} = -\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_{22}^{22} & c_{22}^{22} & 0 & 0 \\ c_{31}^{22} & 0 & c_{33}^{22} & 0 \\ c_{41}^{22} & c_{42}^{22} & c_{43}^{22} & c_{44}^{22} \end{bmatrix}$$

$$c_{21}^{11} = -(2\mu + \lambda)\frac{m_1}{\rho^2}, \qquad c_{22}^{11} = (2\mu + \lambda)\frac{1}{\rho}, \qquad c_{31}^{11} = -\mu\frac{m_2}{\rho^2}, \qquad c_{33}^{11} = \frac{\mu}{\rho},$$

$$c_{41}^{11} = -(2\mu + \lambda)\frac{m_1^2}{\rho^3} - \mu\frac{m_2^2}{\rho^3} + \frac{k}{c_v}\left(-\frac{e}{\rho^2} + \frac{m_1^2 + m_2^2}{\rho^3}\right),$$

$$c_{42}^{11} = \left(2\mu + \lambda - \frac{k}{c_v}\right)\frac{m_1}{\rho^2}, \qquad c_{43}^{11} = \left(\mu - \frac{k}{c_v}\right)\frac{m_2}{\rho^2}, \qquad c_{44}^{11} = \frac{k}{c_v}\frac{1}{\rho},$$

$$c_{21}^{12} = -\lambda\frac{m_2}{\rho^2}, \quad c_{23}^{12} = \frac{\lambda}{\rho}, \qquad c_{31}^{12} = -\mu\frac{m_1}{\rho^2}, \qquad c_{32}^{12} = \frac{\mu}{\rho},$$

$$c_{41}^{12} = -(\mu + \lambda)\frac{m_1 m_2}{\rho^3}, \qquad c_{42}^{12} = \mu\frac{m_2}{\rho^2}, \qquad c_{43}^{12} = \lambda\frac{m_1}{\rho^2},$$

$$c_{21}^{21} = -\mu\frac{m_2}{\rho^2}, \qquad c_{23}^{21} = \frac{\mu}{\rho}, \qquad c_{31}^{21} = -\lambda\frac{m_1}{\rho^2}, \qquad c_{32}^{21} = \frac{\lambda}{\rho}$$

$$c_{41}^{21} = -(\mu + \lambda)\frac{m_1 m_2}{\rho^3}, \qquad c_{42}^{21} = \lambda\frac{m_2}{\rho^2}, \qquad c_{43}^{21} = \mu\frac{m_1}{\rho^2},$$

$$c_{21}^{22} = -\mu\frac{m_1}{\rho^2}, \qquad c_{22}^{22} = \frac{\mu}{\rho}, \qquad c_{31}^{22} = -(2\mu + \lambda)\frac{m_2}{\rho^2}, \qquad c_{33}^{22} = (2\mu + \lambda)\frac{1}{\rho},$$

$$c_{41}^{22} = -(2\mu + \lambda)\frac{m_2^2}{\rho^3} - \mu\frac{m_1^2}{\rho^3} + \frac{k}{c_v}\left(-\frac{e}{\rho^2} + \frac{m_1^2 + m_2^2}{\rho^3}\right),$$

$$c_{42}^{22} = \left(\mu - \frac{k}{c_v}\right)\frac{m_1}{\rho^2}, \qquad c_{43}^{22} = \left(2\mu + \lambda - \frac{k}{c_v}\right)\frac{m_2}{\rho^2}, \qquad c_{44}^{22} = \frac{k}{c_v}\frac{1}{\rho}$$

A typical implicit scheme may be constructed by linearizing the convection flux, diffusion flux, and diffusion gradient as follows:

$$\mathbf{F}_i^{n+1} = \mathbf{F}_i^n + \frac{\partial \mathbf{F}_i^n}{\partial \mathbf{U}} \Delta \mathbf{U}^{n+1} = \mathbf{F}_i^n + \mathbf{a}_i^n \Delta \mathbf{U}^{n+1}$$

$$\mathbf{G}_i^{n+1} = \mathbf{G}_i^n + \frac{\partial \mathbf{G}_i^n}{\partial \mathbf{U}} \Delta \mathbf{U}^{n+1} + \frac{\partial \mathbf{G}_i^n}{\partial \mathbf{U}_{,j}} \Delta \mathbf{U}^{n+1} = \mathbf{G}_i^n + \mathbf{b}_i^n \Delta \mathbf{U}^{n+1} + \mathbf{c}_{ij}^n \Delta \mathbf{U}_{,j}^{n+1}$$

$$\frac{\Delta \mathbf{U}^{n+1}}{\Delta t} = -\frac{1}{2}\left[\left(\frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i}\right)^n + \left(\frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i}\right)^{n+1}\right]$$

using the relation, $\mathbf{c}_{ij} \Delta \mathbf{U}_{,j} = (\mathbf{c}_{ij} \Delta \mathbf{U})_{,j} - \mathbf{c}_{ij,j} \Delta \mathbf{U}$

it follows that

$$\left\{\mathbf{I} + \frac{\Delta t}{2}\left[\frac{\partial}{\partial x_i}(\mathbf{a}_i + \mathbf{b}_i - \mathbf{c}_{ij,j}) + \frac{\partial^2 \mathbf{c}_{ij}}{\partial x_i \partial x_j}\right]^n\right\} \Delta \mathbf{U}^{n+1} = -\Delta t \left(\frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i}\right)^n$$

For simplicity of notation, let the Navier-Stokes system of equations be written as

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{W}, \qquad \mathbf{W} = -\frac{\partial \mathbf{F}_i}{\partial x_i} - \frac{\partial \mathbf{G}_i}{\partial x_i}$$

The Beam-Warming implicit method begins with an introduction of implicitness parameters ξ and θ such that

$$\frac{1}{\Delta t}\left[(1+\xi)\Delta \mathbf{U}^{n+1} - \xi \Delta \mathbf{U}^{n}\right] = \theta \mathbf{W}^{n+1} + (1-\theta)\mathbf{W}^{n}$$

with $0 \leq (\xi, \theta) \leq 1$, $\Delta \mathbf{U}^{n+1} = \mathbf{U}^{n+1} - \mathbf{U}^{n}$, and $\Delta \mathbf{U}^{n} = \mathbf{U}^{n} - \mathbf{U}^{n-1}$.

$$\Delta \mathbf{U}^{n+1} = \frac{\Delta t}{1+\xi}\left[\frac{\partial}{\partial t}(\theta \Delta \mathbf{U}^{n+1} + \mathbf{U}^{n}) + \xi \frac{\Delta \mathbf{U}^{n}}{\Delta t}\right]$$

$$\frac{1}{\Delta t}[(1+\xi)\Delta\mathbf{U}^{n+1} - \xi\,\Delta\mathbf{U}^{n}] = -\theta\left[\frac{\partial}{\partial x_i}(\mathbf{a}_i\,\Delta\mathbf{U} + \mathbf{b}_i\,\Delta\mathbf{U} + \mathbf{c}_{ij}\,\Delta\mathbf{U}_{,j})\right]^{n+1} - \left[\frac{\partial}{\partial x_i}(\mathbf{F}_i + \mathbf{G}_i)\right]^{n}$$

or

$$\left\{\mathbf{I} + \frac{\theta\,\Delta t}{1+\xi}\left[\frac{\partial}{\partial x_i}(\mathbf{a}_i + \mathbf{b}_i - \mathbf{c}_{ij,j}) + \frac{\partial^2 \mathbf{c}_{ij}}{\partial x_i \partial x_j}\right]^{n}\right\}\Delta\mathbf{U}^{n+1} = \frac{\xi}{1+\xi}\Delta\mathbf{U}^{n} - \frac{\Delta t}{1+\xi}\left(\frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i}\right)^{n}$$

rewritten as

$$\left\{\mathbf{I} + \frac{\theta\,\Delta t}{1+\xi}\left[\frac{\partial}{\partial x_i}(\mathbf{a}_i + \mathbf{b}_i - \mathbf{c}_{ij,j}) + \frac{\partial^2 \mathbf{c}_{ij}}{\partial x_i \partial x_j}\right]^{n}\right\}\Delta\mathbf{U}^{n+1} = \frac{\xi}{1+\xi}\Delta\mathbf{U}^{n} - \frac{\Delta t}{1+\xi}\left(\frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i}\right)^{n} + \frac{\Delta t\,\theta}{1+\xi}\,\frac{\partial \mathbf{G}_{(i)}^{n-1}}{\partial x_{(j)}}$$

$$\left\{\mathbf{I}+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x}(\mathbf{a}_1+\mathbf{b}_1-\mathbf{c}_{11.1})+\frac{\partial^2\mathbf{c}_{11}}{\partial x^2}+\frac{\partial}{\partial y}(\mathbf{a}_2+\mathbf{b}_2-\mathbf{c}_{22.2})+\frac{\partial^2\mathbf{c}_{22}}{\partial y^2}\right]^n\right\}\Delta\mathbf{U}^{n+1}$$

$$=\text{RHS} \qquad (6.3.1$$

$$\text{RHS}=\frac{\xi}{1+\xi}\Delta\mathbf{U}^n-\frac{\Delta t}{1+\xi}\mathbf{W}^n+\frac{\theta\Delta t}{1+\xi}\left[\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}\right]^n$$

$$+O\left[\left(\theta-\frac{1}{2}-\xi\right)\Delta t^2,\Delta t^3\right]$$

with

$$\mathbf{b}_1-\mathbf{c}_{11.1}=\frac{1}{\rho}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -u\left(\frac{4}{3}\mu\right)_x & \left(\frac{4}{3}\mu\right)_x & 0 & 0 \\ -v\mu_x & 0 & \mu_x & 0 \\ -u^2\left(\frac{4}{3}\mu\right)_x-v^2\mu_x & u\left(\frac{4}{3}\mu\right)_x & v\mu_x & 0 \end{bmatrix}$$

$$\mathbf{b}_2 - \mathbf{c}_{22.2} = \frac{1}{\rho}\begin{bmatrix} 0 & 0 & 0 & 0 \\ -u\mu_x & \mu_y & 0 & 0 \\ -v\left(\frac{4}{3}\mu\right)_y & 0 & \left(\frac{4}{3}\mu\right)_y & 0 \\ -v^2\left(\frac{4}{3}\mu\right)_y - u^2\mu_y & u\mu_y & v\frac{4}{3}\mu_y & 0 \end{bmatrix}$$

Step 1

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial x}(\mathbf{a}_1 + \mathbf{b}_1 - \mathbf{c}_{11.1}) + \frac{\partial^2 \mathbf{c}_{11}}{\partial x^2}\right]^n\right\}\Delta\mathbf{U}^* = \text{RHS}$$

Step 2

$$\left\{\mathbf{I} + \frac{\theta\Delta t}{1+\xi}\left[\frac{\partial}{\partial y}(\mathbf{a}_2 + \mathbf{b}_2 - \mathbf{c}_{22.2}) + \frac{\partial^2 \mathbf{c}_{22}}{\partial y^2}\right]^n\right\}\Delta\mathbf{U}^{n+1} = \Delta\mathbf{U}^*$$

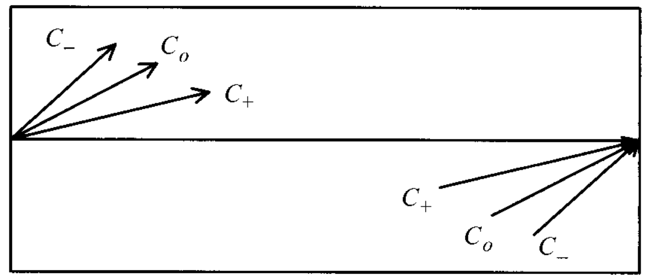
Characteristic Boundary Conditions

$$\Delta \mathbf{W} = \begin{bmatrix} \Delta \mathbf{W}_a \\ \Delta \mathbf{W}_b \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{aa}^{-1} & \mathbf{L}_{ab}^{-1} \\ \mathbf{L}_{ba}^{-1} & \mathbf{L}_{bb}^{-1} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{V}_a \\ \Delta \mathbf{V}_b \end{bmatrix} = \begin{bmatrix} -1/\rho a & 0 & 1 \\ -1/a^2 & 1 & 0 \\ 1/\rho a & 0 & 1 \end{bmatrix}_0 \begin{bmatrix} \Delta p \\ \Delta \rho \\ \Delta u \end{bmatrix}$$

Table 6.7.1 Physical and Numerical Boundary Conditions

		Subsonic	Supersonic
Inlet	Physical	W_1, W_2	W_1, W_2, W_3
	Numerical	W_3	None
Outlet	Physical	W_3	None
	Numerical	W_1, W_2	W_1, W_2, W_3

Supersonic Inflow | Supersonic Outflow



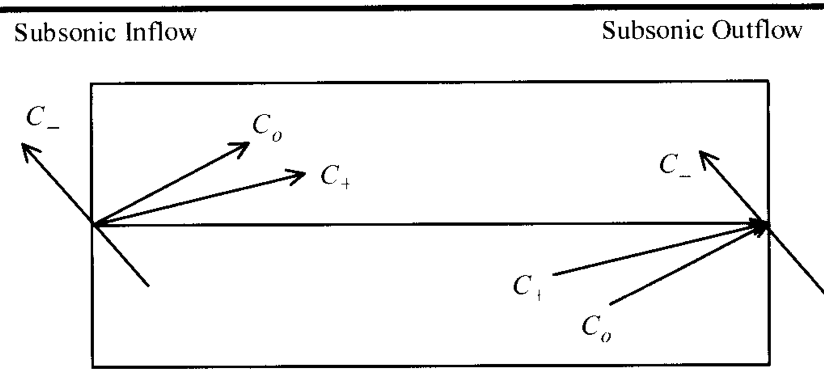
Three characteristics going into the domain. (Three physical boundary conditions required)

Three characteristics going out of the domain. (No physical boundary conditions. Instead, three numerical boundary conditions required)

	INFLOW		OUTFLOW	
	Euler	Navier-Stokes	Euler	Navier-Stokes
1-D	$\rho, \rho v_i, \rho E \quad (i=1)$		Dirichlet, none (Set Free) All variables extrapolated	
2-D	$\rho, \rho v_i, \rho E \quad (i=1,2)$	$\rho, \rho v_i, \rho E \quad (i=1,2)$	Dirichlet, none (Set Free) $\rho E_{,i} n_i = 0$	Dirichlet, none (Set Free) $\tau_{ij} n_j = 0 \quad (i=1,2)$ $\rho E_{,i} n_i = 0$
3-D	$\rho, \rho v_i, \rho E \quad (i=1,2,3)$	$\rho, \rho v_i, \rho E \quad (i=1,2,3)$	Dirichlet, none (Set Free) $\rho E_{,i} n_i = 0$	Dirichlet none (Set Free) $\tau_{ij} n_j = 0 \quad (i=1,2,3)$ $\rho E_{,i} n_i = 0$

(a)

Figure 6.7.1 Summary of boundary conditions for compressible flows. **(a)** Supersonic flow boundary conditions. **(b)** Subsonic flow boundary conditions.

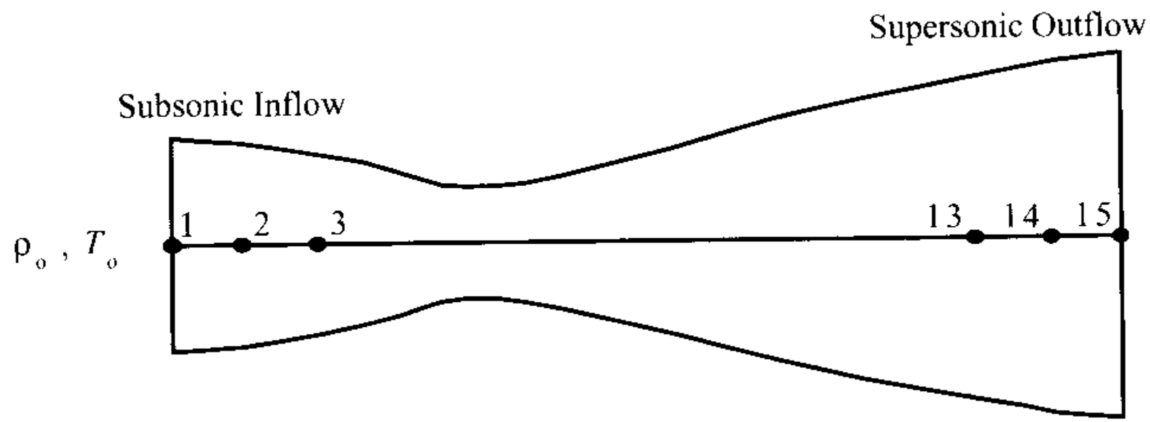


Two characteristics going into the domain, one characteristic going out of the domain. (One numerical and two physical boundary conditions required)

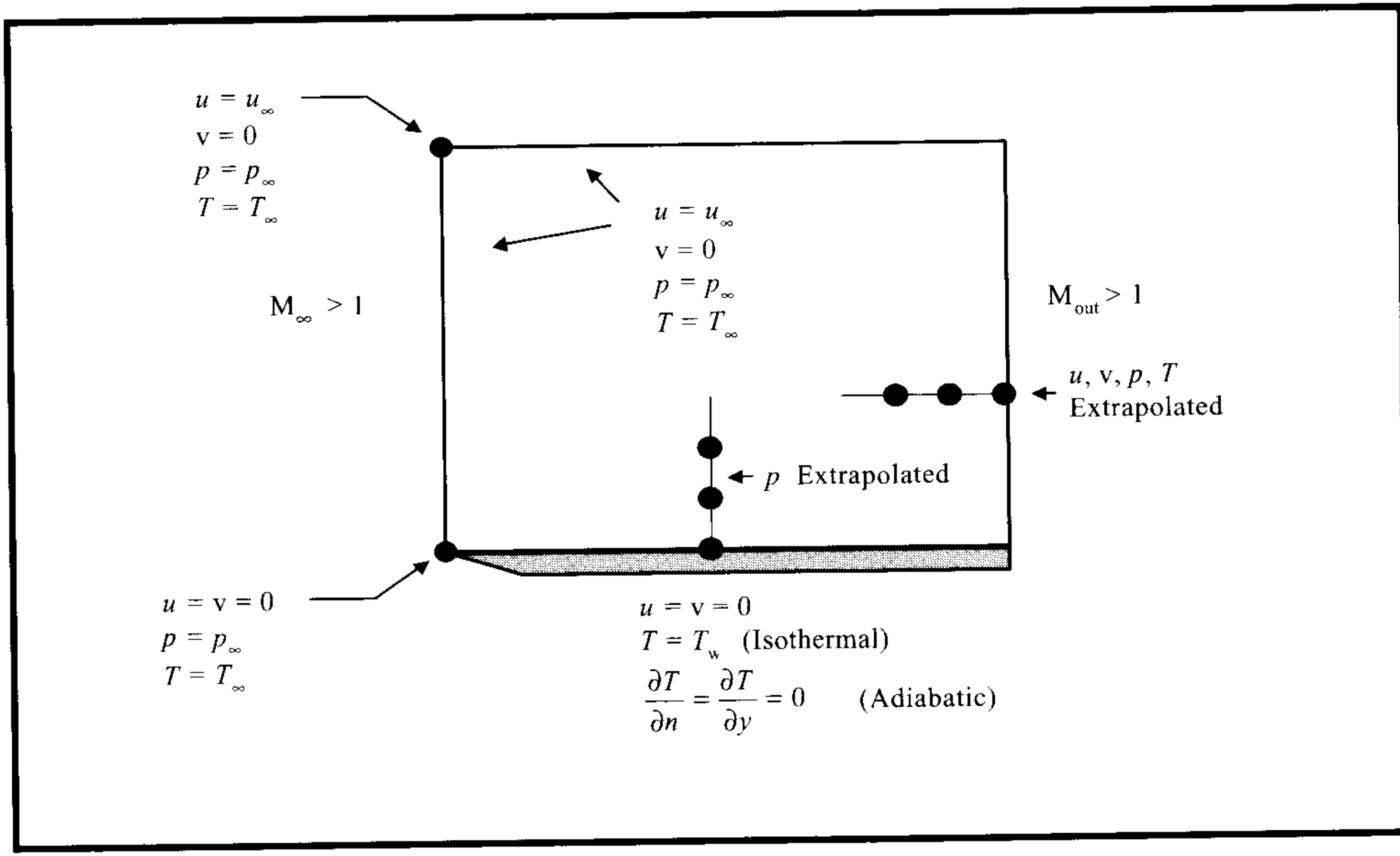
Two characteristics going out of the domain, one characteristic going into the domain. (One physical and two numerical boundary conditions required)

	INFLOW		OUTFLOW	
	Euler	Navier-Stokes	Euler	Navier-Stokes
1-D	$\rho, \rho E$ ρv_i extrapolated		$\rho v_i \quad (i=1)$ All other variables extrapolated	
2-D	$\rho, \rho E$ ρv_i extrapolated	$\rho, \rho E$ ρv_i extrapolated	$\rho v_i \quad (i=1,2)$ All other variables extrapolated	$\rho v_i \quad (i=1,2)$ $\tau_{ij} n_j = 0 \quad (i=1,2)$ $\rho E_{,i} n_i = 0$
3-D	$\rho, \rho E$ ρv_i extrapolated	$\rho, \rho E$ ρv_i extrapolated	$\rho v_i \quad (i=1,2,3)$ All other variables extrapolated	$\rho v_i \quad (i=1,2,3)$ $\rho E_{,i} n_i = 0$

(b)



INFLOW	OUTFLOW
Physical boundary conditions, (ρ_1 , T_1)	No physical boundary conditions
Numerical boundary conditions Extrapolate: $u_1 = 2u_2 - u_3$ Compute: $p = \rho RT$	Numerical boundary conditions Extrapolate: $u_{15} = 2u_{14} - u_{13}$, $\rho_{15} = 2\rho_{14} - \rho_{13}$, $T_{15} = 2T_{14} - T_{13}$
Initial conditions, $\rho(x,0) = T(x,0) = u(x,0) = 0$; If outflow is subsonic, specify p_{15} and set the others free.	



$u = u_\infty$
$v = 0$
$p = p_\infty$
$T = T_\infty$
$u = u_\infty$
$v = 0$
$p = p_\infty$
$T = T_\infty$
$M_\infty > 1$
$M_{out} > 1$
u, v, p, T
Extrapolated
p Extrapolated
$u = v = 0$
$p = p_\infty$
$T = T_\infty$
$u = v = 0$
$T = T_w$ (Isothermal)
$\frac{\partial T}{\partial n} = \frac{\partial T}{\partial y} = 0$ (Adiabatic)

(b)